VII. *The Extinction of Sound in a Viscous Atmosphere by Small Obstacles of Cylindrical and Spherical Form.*

By C. J. T. Sewell, *B.A.*, *Trinity College, Cambridge.*

Communicated by Prof. H. Lamb, *F.R.S.*

Received February 1,—Read March 10, 1910.

Contents.

		Page
§ 1.	Expression obtained for the dissipation function in the case of aerial vibrations	241
§ 2.	Solution of the equations of motion with reference to cylindrical surfaces	242
§ 3.	Incidence of plane waves of sound upon obstructing cylinder. An expression is obtained for the secondary waves	244
§ 4.	Determination of the lost energy. Discussion of results	248
§ 5.	Incidence of plane waves of sound upon a large number of cylindrical obstacles	254
§ 6.	Problems relating to spherical obstacles. A solution of the equations of motion suitable to such problems is obtained	257
§ 7.	Incidence of plane waves of sound upon obstructing sphere. An expression is obtained for the secondary waves	259
§ 8.	Determination of the lost energy. Discussion of results	262
§ 9.	Incidence of plane waves of sound upon a large number of spherical obstacles	265

Introduction.—The theory of the incidence of waves of sound in a non-viscous air upon small obstacles of cylindrical or spherical form is well known to students of mathematical physics; it has been treated in Lord Rayleigh's 'Theory of Sound,' and in Prof. Lamb's 'Treatise on Hydrodynamics.' The corresponding problems for a viscous air have not, however, been worked out, and this paper is devoted to an investigation of these problems. The solutions of the equations of vibration of a viscous gas with reference to cylindrical and spherical surfaces were given by Prof. Lamb in a paper entitled "On the Motion of a Viscous Fluid Contained in a Spherical Vessel" and published in the 'Proceedings of the London Mathematical Society' in 1884. It is easy to obtain solutions suitable to the case of divergent waves; the functions involved are Bessel functions with a complex argument. An analytical expression for the secondary waves diverging from the obstacle is obtained without difficulty. It then remains to find an expression for the loss of energy to the primary waves. In calculating this loss of energy it is necessary to consider the dissipation of energy by friction in the immediate neighbourhood of the obstacle in addition to the energy which is carried away to a distance by the secondary waves. This was pointed out to me by Prof. LAMB, at whose suggestion this paper was written. In obtaining an expression for the energy dissipated by friction I at first made use of the dissipation function. This method led to exactly the same results as that finally adopted, but the mathematics involved were cumbrous, and the physical ideas, on which they were based, were somewhat obscure. Another disadvantage of this method was that it was necessary to calculate separately the scattered and the dissipated energy. I have to thank Prof. LAMB for his kindness in pointing out to me the method of calculating the lost energy adopted in this paper. The result has been to make the paper more clear and readable.

I have succeeded in obtaining expressions for the energy lost to the primary waves in the case of spherical and cylindrical obstacles. As might be expected, the problem of the cylindrical obstacle presents greater analytical difficulty than that of the spherical obstacle, and in the former case it is necessary to obtain different approximate expressions according to the diameter of the obstacle. The results for wires of 10^{-1} cm. radius and for wires of 10^{-3} cm. radius can be obtained without much difficulty, but when the radius of the wire is of order 10^{-2} cm. it is necessary to perform very laborious calculations in order to arrive at intelligible results. The energy lost to the primary waves is, in all cases, very great compared with what would be lost in a non-viscous air, but the ratio of the lost energy to that incident upon the obstacle is at most of order 10^{-2}.

In the case of spherical obstacles the difficulties of approximation are not so great, as in the case of cylindrical obstacles the loss of energy is far greater than in a non-viscous air, but, as before, the ratio of the lost energy to that incident upon the obstacle is at most of order 10^{-2}.

It is possible to extend the results obtained for a single obstacle to the case when the waves of sound are incident upon a large number of similar obstacles. This has been done by Lord RAYLEIGH for the corresponding problem in a non-viscous air; the same method has been adopted in this paper. It should, however, be borne in mind that the results so obtained are valid only when the obstacles are so sparsely distributed that the space occupied by the obstacles is a small fraction of the total volume. The investigation has some practical interest. It has been asserted that the suspension of a large number of parallel wires in a hall or lecture room will improve the acoustic properties of the room. According to the ordinary theory of a non-viscous air the effect of any such arrangement of wires would be inappreciable. From the results of this paper it also appears that the viscosity of the air is not sufficient to account for the alleged phenomenon.

The results in the case of spherical obstacles are of greater interest, since they are applicable to the consideration of the effect of foggy weather upon the propagation and audibility of sound. If the diameter of the drops of water in a dense fog is assumed to be ·02 mm., there is no appreciable alteration in the audibility of sound; but, if the diameter of the drops of water is ·002 mm., the presence of fog is distinctly prejudicial to the audibility of sound. The former case is in agreement with TYNDALL's observations on the subject.

In conclusion I desire to thank Prof. LAMB for much kind advice and encouragement in the writing of this paper.

§ 1. In a viscous gas, if u, v, w be the components of the velocity at any point x, y, z of the fluid referred to fixed rectangular axes, and if p be the pressure at this point, the equations of vibration may be written in the form

$$\left.\begin{aligned}\frac{\partial u}{\partial t} &= -\frac{1}{\rho_0}\frac{\partial p}{\partial x}+\nu\nabla^2 u+\tfrac{1}{3}\nu\frac{\partial \vartheta}{\partial x}\\ \frac{\partial v}{\partial t} &= -\frac{1}{\rho_0}\frac{\partial p}{\partial y}+\nu\nabla^2 v+\tfrac{1}{3}\nu\frac{\partial \vartheta}{\partial y}\\ \frac{\partial w}{\partial t} &= -\frac{1}{\rho_0}\frac{\partial p}{\partial z}+\nu\nabla^2 w+\tfrac{1}{3}\nu\frac{\partial \vartheta}{\partial z}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where ρ_0 is the equilibrium density, ν is a small constant of dimensions L^2T^{-1} depending on the viscosity, ϑ has been written for *div* (u, v, w), and terms of the order of the square of the velocity have been neglected.

By a method very similar to that* used in the case of an incompressible fluid it is found that the total rate of dissipation of energy within any closed surface S is given by

$$\begin{aligned}2F = {}&\tfrac{4}{3}\nu\rho_0\iiint(a+b+c)^2\,dx\,dy\,dz+4\nu\rho_0\iiint(\xi^2+\eta^2+\zeta^2)\,dx\,dy\,dz\\ &+2\nu\rho_0\iint(lu+mv+nw)\,(a+b+c)\,dS-\nu\rho_0\iint\frac{dq^2}{dn}dS+4\nu\rho_0\iint\begin{vmatrix} l, & m, & n\\ u, & v, & w\\ \xi, & \eta, & \zeta\end{vmatrix}dS \quad . \quad (2)\end{aligned}$$

where q is the resultant velocity at any point of the fluid, δn denotes an element of the normal to the surface S, l, m, n are the direction cosines of this normal drawn inwards in each case from the surface element δS. Further, a, b, c, ξ, η, ζ are given by the relations

$$\left.\begin{aligned}&a=\frac{\partial u}{\partial x}, \qquad b=\frac{\partial v}{\partial y}, \qquad c=\frac{\partial w}{\partial z}\\ &2\xi=\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}, \quad 2\eta=\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}, \quad 2\zeta=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (3).$$

When there is no motion of the gas parallel to the axis of z, and the motion is the same in all planes perpendicular to this direction, the expression (2) for the dissipation of energy per unit length of the axis of z takes the form

---

* LAMB's 'Hydrodynamics,' p. 540.

$$2\mathrm{F} = \tfrac{4}{3}\nu\rho_0 \iint (a+b)^2\,dx\,dy + 4\nu\rho_0 \iint \zeta^2\,dx\,dy$$
$$+2\nu\rho_0 \int (lu+mv)\,(a+b)\,ds - \nu\rho_0 \int \frac{\partial q^2}{\partial n}\,ds + 4\nu\rho_0 \int (lv-mu)\,\zeta\,ds \quad . \quad . \quad . \quad (4)$$

where ds is an element of the curve bounding the region in question.

§ 2. We now proceed to obtain a solution of the equations of motion which shall be applicable to the case when the motion in all planes perpendicular to the axis of z is the same, and when further there is no motion parallel to this axis.

In this case the equations of motion take the form

$$\left.\begin{aligned} \frac{\partial u}{\partial t} &= -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \nu\nabla_1{}^2 u + \tfrac{1}{3}\nu\frac{\partial \vartheta}{\partial x} \\ \frac{\partial v}{\partial t} &= -\frac{1}{\rho_0}\frac{\partial p}{\partial y} + \nu\nabla_1{}^2 v + \tfrac{1}{3}\nu\frac{\partial \vartheta}{\partial y} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

where $\vartheta = \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y}$, and $\nabla_1{}^2$ denotes the operator $\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$.

The equation of continuity takes the form

$$\frac{\partial s}{\partial t} + \vartheta = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2),$$

where squares of the velocity and other quantities of the same order have been ignored, and s denotes the condensation.

If we neglect the effects of conduction and radiation of heat, we may write

$$p = p_0 + c^2\rho_0 s \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$$

where p_0 is the equilibrium pressure and c is the velocity of sound.

Eliminating p and s from equations (1) with the help of (2) and (3), we obtain

$$\left.\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= c^2\frac{\partial \vartheta}{\partial x} + \nu\nabla_1{}^2\frac{\partial u}{\partial t} + \tfrac{1}{3}\nu\frac{\partial^2 \vartheta}{\partial x\,\partial t} \\ \frac{\partial^2 v}{\partial t^2} &= c^2\frac{\partial \vartheta}{\partial y} + \nu\nabla_1{}^2\frac{\partial v}{\partial t} + \tfrac{1}{3}\nu\frac{\partial^2 \vartheta}{\partial y\,\partial t} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

These equations will be satisfied by

$$u = \frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y}, \qquad v = \frac{\partial \phi}{\partial y} - \frac{\partial \psi}{\partial x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5),$$

provided ϕ and ψ are functions satisfying the equations

$$\frac{\partial^2\phi}{\partial t^2} = c^2\nabla_1{}^2\phi + \tfrac{4}{3}\nu\frac{\partial}{\partial t}\nabla_1{}^2\phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6),$$

$$\frac{\partial\psi}{\partial t} = \nu\nabla_1{}^2\psi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7).$$

If we assume a time factor $e^{\iota\sigma t}$, these equations take the forms

$$(\nabla_1{}^2+h^2)\,\phi = 0, \qquad (\nabla_1{}^2+k^2)\,\psi = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8),$$

where h^2 and k^2 are given by

$$h^2 = \sigma^2/(c^2+\tfrac{4}{3}\iota\nu\sigma), \qquad k^2 = -\iota\sigma/\nu \quad . \quad . \quad . \quad . \quad . \quad (9),\ (10).$$

We shall for convenience suppose h to be equal to that root of equation (9) which reduces to σ/c when ν is zero; k will be taken to be equal to $(\sigma/\nu)^{1/2}.e^{-\iota\pi/4}$.

With these conventions the solution of the equations (8), which represents waves of sound diverging from the origin, is given by

$$\left.\begin{aligned} \phi &= \mathrm{A}_0\mathrm{D}_0(hr) + \sum_{n=1}^{\infty} \mathrm{A}_n\mathrm{D}_n(hr)\cos(n\vartheta+\alpha_n) \\ \psi &= \mathrm{B}_0\mathrm{D}_0(kr) + \sum_{n=1}^{\infty} \mathrm{B}_n\mathrm{D}_n(kr)\sin(n\vartheta+\beta_n) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (11),$$

where for convenience the time factor has been omitted and where $\mathrm{D}_n(\zeta)$ is given by

$$\mathrm{D}_n(\zeta) = \frac{2}{\pi}\{\log 2-\gamma-\tfrac{1}{2}\iota\pi)\,\mathrm{J}_n(\zeta)-\mathrm{Y}_n(\zeta)\},$$

$\mathrm{J}_n(\zeta)$ and $\mathrm{Y}_n(\zeta)$ have their usual significance as Bessel functions, and A_n, B_n, α_n, and β_n are arbitrary constants.

It need hardly be remarked that to obtain the actual expressions for ϕ and ψ it is necessary to multiply the expressions contained in (11) by $e^{\iota\sigma t}$, and to equate ϕ and ψ to the real parts only of these products. For the sake of brevity we shall usually omit, when possible, the time factor $e^{\iota\sigma t}$.

Since, in the case of air, ν is a small quantity, when expressed in cm. sec. units, it is clear from (9) and (10) that for all audible sounds $|k|$ will be large compared with $|h|$. In fact, at any ordinary distance r from the origin the ψ terms in (11) will become insensible owing to the factor exp. $\{-(\tfrac{1}{2}\sigma/\nu)^{1/2}r\}$.

For plane waves propagated in the negative direction of the axis of x, the solution will be given by

$$\phi = \mathrm{C}e^{\iota hx}e^{\iota\sigma t}, \qquad \psi = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12),$$

where, as before, only the real part of ϕ is to be taken into account.

§ 3. *The Incidence of Plane Waves of Sound upon an Obstructing Cylinder.*—We are now in a position to consider the effect of a cylindrical obstacle upon a train of waves propagated in a direction perpendicular to the axis of the obstacle and incident upon it.

We take the axis of the obstructing cylinder as axis of z, and suppose the incident sound to be propagated in the negative direction along the axis of x. Then, as in (12) of the last article, we may assume for the incident sound the expressions

$$\phi_0 = e^{\iota hx}, \qquad \psi_0 = 0.$$

Expanding in series of Bessel functions, we obtain

$$\phi_0 = \mathrm{J}_0(hr) + \sum_{n=1}^{\infty} 2\iota^n \mathrm{J}_n(hr)\cos n\vartheta, \qquad \psi_0 = 0 \quad . \quad . \quad . \quad . \quad . \quad (1).$$

The scattered waves will be symmetrical about the axis of x, or $\vartheta = 0$, and consequently we may assume for them the forms

$$\phi_1 = \mathrm{A}_0\,\mathrm{D}_0(hr) + \sum_{n=1}^{\infty}\{\mathrm{A}_n\,\mathrm{D}_n(hr)\cos n\vartheta\},$$

$$\psi_1 = \sum_{n=1}^{\infty}\{\mathrm{B}_n\,\mathrm{D}_n(kr)\sin n\vartheta\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

At the surface of the obstacle the radial and tangential components of the velocity must vanish; hence we must have

$$r\frac{\partial(\phi_0+\phi_1)}{\partial r} + \frac{\partial\psi_1}{\partial\vartheta} = 0, \qquad -\frac{\partial(\phi_0+\phi_1)}{\partial\vartheta} + r\frac{\partial\psi_1}{\partial r} = 0 \quad . \quad . \quad . \quad . \quad (3)$$

when $r = a$, if a is the radius of the cylinder.

In order that the boundary conditions (3) may be satisfied, we must have

$$\mathrm{A}_0 ha\,\mathrm{D}_0'(ha) = -ha\mathrm{J}_0'(ha)$$

or

$$\mathrm{A}_0\,\mathrm{D}_1(ha) = -\mathrm{J}_1(ha) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4),$$

and in general for $n > 0$

$$\left.\begin{aligned} \mathrm{A}_n ha\,\mathrm{D}_n'(ha) + n\mathrm{B}_n\,\mathrm{D}_n(ka) &= -2\iota^n ha\mathrm{J}_n'(ha) \\ n\mathrm{A}_n\,\mathrm{D}_n(ha) + \mathrm{B}_n ka\,\mathrm{D}_n'(ka) &= -2\iota^n n\mathrm{J}_n(ha) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

These equations (4) and (5) are sufficient to determine the various constants in the expressions (2) for the scattered sound. In the process of approximating to the values of these constants by means of equations (4) and (5), we shall confine ourselves to the case when ha is a small quantity; in other words, we shall assume that the dimensions of the obstacle are small compared with the wave-length of the incident sound. The other case, when the dimensions of the obstacle are large in comparison with the wave-length, presents exactly the same difficulties as occur in the similar problem in connection with the theory of a non-viscous gas.

We shall also consider especially the case when the viscous gas is the air of the atmosphere; in this case ν is a small quantity about ·132 in cm. sec. units, and consequently we may regard $\sigma\nu/c^2$ as a small quantity for all wave-lengths. Since $\sigma\nu/c^2$ is small, we may write very approximately from (9), § 2,

$$h = \frac{\sigma}{c}\left(1-\tfrac{2}{3}\iota\sigma\nu/c^2\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6).$$

We must now obtain approximations to the values of the constants in the expressions for the scattered sound by means of equations (4) and (5). For this we shall need the approximate values of the Bessel functions involved in these equations. For convenience we shall write them down.

When ζ is small, we have for all values of $n>0$,

$$\left.\begin{aligned}
\mathrm{D}_n(\zeta) &= \frac{2^n(n-1)!}{\pi}\zeta^{-n} + \text{less important terms} \\
\mathrm{D}_n'(\zeta) &= -\frac{2^n n!}{\pi}\zeta^{-(n+1)} + \ldots \\
\mathrm{J}_n(\zeta) &= \frac{1}{2^n n!}\zeta^n + \text{terms containing higher powers of } \zeta \\
\mathrm{J}_n'(\zeta) &= \frac{1}{2^n . (n-1)!}\zeta^{n-1} + \text{terms containing higher powers of } \zeta
\end{aligned}\right\} \quad . \quad . \quad (7).$$

Closer approximations are

$$\left.\begin{aligned}
&\mathrm{D}_0(\zeta) = \frac{2}{\pi}\left\{-\left(\log\tfrac{1}{2}\zeta+\gamma+\tfrac{1}{2}\iota\pi\right)+\tfrac{1}{2}\zeta^2\left(\log\tfrac{1}{2}\zeta+\gamma+\tfrac{1}{2}\iota\pi-\tfrac{1}{2}\right)\right\} \\
&\mathrm{D}_1(\zeta) = \frac{2}{\pi}\left\{\zeta^{-1}-\tfrac{1}{2}\zeta\left(\log\tfrac{1}{2}\zeta+\gamma+\tfrac{1}{2}\iota\pi-\tfrac{1}{2}\right)\right\}, \qquad \mathrm{D}_2(\zeta) = \frac{2}{\pi}\left(2\zeta^{-2}+\tfrac{1}{2}\right)
\end{aligned}\right\} \quad . \quad (8).$$

Also we have

$$\mathrm{J}_0(\zeta) = 1-\tfrac{1}{4}\zeta^2, \qquad \mathrm{J}_1(\zeta) = \tfrac{1}{2}\zeta\left(1-\tfrac{1}{8}\zeta^2\right), \qquad \mathrm{J}_2(\zeta) = \tfrac{1}{8}\zeta^2\left(1-\tfrac{1}{12}\zeta^2\right) \quad . \quad . \quad (9).$$

Now, on eliminating A_n from equations (5), we obtain

$$\begin{aligned}
\mathrm{B}_n\left[ha\,.\,ka\,\mathrm{D}_n'(ha)\,\mathrm{D}_n'(ka)-n^2\,\mathrm{D}_n(ha)\,\mathrm{D}_n(ka)\right]& \\
= -2\iota^n nha\left\{\mathrm{J}_n(ha)\,\mathrm{D}_n'(ha)-\mathrm{D}_n(ha)\,\mathrm{J}_n'(ha)\right\}&.
\end{aligned}$$

Further

$$\mathrm{J}_n(\zeta)\,\mathrm{D}_n'(\zeta)-\mathrm{D}_n(\zeta)\,\mathrm{J}_n'(\zeta) = \frac{2}{\pi}\left\{\mathrm{Y}_n(\zeta)\,\mathrm{J}_n'(\zeta)-\mathrm{Y}_n'(\zeta)\,\mathrm{J}_n(\zeta)\right\} = -\frac{2}{\pi}\zeta^{-1}$$

by a well known result in the theory of Bessel functions.

Consequently we obtain

$$\mathrm{B}_n\,[ha\,.\,ka\,\mathrm{D}_n'(ha)\,\mathrm{D}_n'(ka)-n^2\,\mathrm{D}_n(ha)\,\mathrm{D}_n(ka)] = \frac{4n}{\pi}\,\iota^n \quad . \quad . \quad . \quad (10).$$

Now ha is a small quantity, and consequently we may use the approximations (7); substituting for $\mathrm{D}_n(ha)$ and $\mathrm{D}_n'(ha)$ from (7), we obtain

$$-\frac{2^n n!}{\pi}\,h^{-n}a^{-n}\mathrm{B}_n\,\{ka\,\mathrm{D}_n'(ka)+n\,\mathrm{D}_n(ka)\} = \frac{4n}{\pi}\,\iota^n$$

or

$$-\frac{2^n n!}{\pi}\,.\,h^{-n}a^{-n}\,.\,ka\,\mathrm{D}_{n-1}(ka)\,\mathrm{B}_n = \frac{4n}{\pi}\,\iota^n.$$

Hence we obtain as a first approximation

$$\mathrm{B}_n = -\frac{2\iota^n}{2^{n-1}}\cdot\frac{1}{(n-1)!}\cdot\frac{h^n a^n}{ka\,\mathrm{D}_{n-1}(ka)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11).$$

Similarly, by elimination of B_n between equations (5), we obtain

$$\mathrm{A}_n\,[ha\,.\,ka\,\mathrm{D}'_n(ha)\,\mathrm{D}'_n(ka)-n^2\,\mathrm{D}_n(ha)\,\mathrm{D}_n(ka)]$$
$$= -2\iota^n\,[ha\,.\,ka\mathrm{J}'_n(ha)\,\mathrm{D}'_n(ka)-n^2\mathrm{J}_n(ha)\,\mathrm{D}_n(ka)].$$

Using the approximate values given in (7), we obtain

$$-\mathrm{A}_n\frac{2^n n!}{\pi}h^{-n}a^{-n}ka\,\mathrm{D}_{n-1}(ka) = 2\iota^n\frac{h^n a^n}{2^n(n-1)!}\,\mathrm{D}_{n+1}(ka)$$

or

$$\mathrm{A}_n = -2\iota^n\,.\,\frac{\pi h^{2n}a^{2n}}{2^{2n}n!\,(n-1)!}\cdot\frac{\mathrm{D}_{n+1}(ka)}{\mathrm{D}_{n-1}(ka)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

for all integral values of $n > 0$.

By a similar process, but carrying the calculation to a higher degree of approximation by means of (8) and (9), we have

$$\mathrm{A}_1 = -\tfrac{1}{2}\iota\pi\,.\,h^2a^2\frac{\mathrm{D}_2(ka)-\frac{1}{8}h^2a^2\{2\mathrm{D}_2(ka)-\mathrm{D}_0(ka)\}}{\mathrm{D}_0(ka)-\frac{1}{2}h^2a^2(\log\frac{1}{2}ha+\gamma+\frac{1}{2}\iota\pi)\,\mathrm{D}_2(ka)+\frac{1}{4}h^2a^2\,\mathrm{D}_0(ka)} \quad . \quad (13).$$

Let us consider first the case when $|ka|$ is small. In this case it will be sufficient to derive the value of A_1 from (12). Hence we have approximately

$$\mathrm{A}_1 = -\tfrac{1}{2}\iota\pi h^2a^2\frac{\mathrm{D}_2(ka)}{\mathrm{D}_0(ka)}.$$

Writing for convenience

$$k = \lambda e^{-1/4\iota\pi}, \quad \text{where} \quad \lambda = (\sigma/\nu)^{1/2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14),$$

and using the approximations for $D_2(\zeta)$ and $D_0(\zeta)$ given in (8), we have

$$D_0(ka) = \frac{2}{\pi}\left[-\left(\log \tfrac{1}{2}\lambda a+\gamma+\tfrac{1}{4}\iota\pi\right)-\tfrac{1}{2}\iota\lambda^2a^2\left(\log \tfrac{1}{2}\lambda a+\gamma+\tfrac{1}{4}\iota\pi-\tfrac{1}{2}\right)\right],$$

$$D_2(ka) = \frac{2}{\pi}\left[2\iota\lambda^{-2}a^{-2}+\tfrac{1}{2}\right].$$

Substituting these expressions for $D_0(ka)$ and $D_2(ka)$ in the above formula for A_1, we obtain

$$A_1 = -\pi\frac{h^2a^2}{\lambda^2a^2}\cdot\frac{1-\frac{1}{4}\iota\lambda^2a^2}{\log \frac{1}{2}\lambda a+\gamma+\frac{1}{4}\iota\pi+\iota\frac{1}{2}\lambda^2a^2\left(\log \frac{1}{2}\lambda a+\gamma+\frac{1}{4}\iota\pi-\frac{1}{2}\right)}.$$

In general, it will tend to sufficient accuracy if we take

$$A_1 = -\frac{\sigma\nu}{c^2}\pi\left\{\log \tfrac{1}{2}\lambda a+\gamma+\tfrac{1}{4}\iota\pi\right\}^{-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

in the case when λa is small.

Similarly we obtain from (11) the approximation

$$B_1 = \iota\pi\frac{ha}{ka}\left(\log \tfrac{1}{2}\lambda a+\gamma+\tfrac{1}{4}\iota\pi\right)^{-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16),$$

when λa is small.

Let us next consider the case when λa is large.

Since $|ka|$ is great, we may write

$$D_0(ka) = \left(\frac{2}{\pi ka}\right)^{1/2} e^{-\iota(ka+1/4\pi)}\left\{1+\frac{\iota}{8ka}+\frac{9\iota^2}{(8ka)^2}\right\},$$

$$D_2(ka) = -\left(\frac{2}{\pi ka}\right)^{1/2} e^{-\iota(ka+1/4\pi)}\left\{1-\frac{15\iota}{8ka}+\frac{105\iota^2}{(8ka)^2}\right\}.$$

Substituting these expressions in the formula given for A_1 in (13), we obtain approximately

$$A_1 \doteqdot \tfrac{1}{2}\iota\pi h^2a^2\left\{1-\frac{16\iota}{8ka}+\frac{112\iota^2}{(8ka)^2}-\tfrac{1}{2}h^2a^2\left(\log \tfrac{1}{2}ha+\gamma+\tfrac{1}{2}\iota\pi+\tfrac{5}{4}\right)\right\}.$$

When $\lambda a e^{-1/4\iota\pi}$ is written for ka, this reduces to

$$\begin{aligned} A_1 = \tfrac{1}{2}\iota\pi h^2a^2\big[1+\surd 2\,.\,(\lambda a)^{-1}-\tfrac{1}{2}h^2a^2\left(\log \tfrac{1}{2}ha+\gamma+\tfrac{5}{4}\right) \\ -\iota\left\{\surd 2\,.\,(\lambda a)^{-1}+\tfrac{7}{4}(\lambda a)^{-2}+\tfrac{1}{4}\pi h^2a^2\right\}\big] \quad . \quad . \quad (17). \end{aligned}$$

As a first approximation to the value of B_1, we find in the case when λa is large

$$B_1 = 2\iota ha\left(\frac{\pi}{2ka}\right)^{1/2} e^{\iota(ka+1/4\pi)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18).$$

The approximate value for A_0, obtained from equation (4) with the help of (8) and (9), is easily seen to be

$$A_0 = -\tfrac{1}{4}\pi h^2a^2\left\{1+\tfrac{1}{2}h^2a^2\left(\log \tfrac{1}{2}ha+\gamma+\tfrac{1}{2}\iota\pi-\tfrac{3}{4}\right)\right\} \quad . \quad . \quad . \quad . \quad . \quad (19).$$

§ 4. Now that we have approximated to the values of the various constants involved in the expressions for the secondary waves, we can proceed to estimate the additional rate at which energy is being dissipated in the space surrounding the obstacle. This additional rate of dissipation will be equivalent to the rate at which energy is being lost to the primary waves in consequence of the presence of the obstacle. Now, if we consider a region bounded internally by the obstacle and externally by a cylindrical surface coaxial with the obstacle and of radius R, it is clear that the difference of the rates at which energy is being carried across the internal and external boundaries of this region will be equivalent to the rate of dissipation of energy within it.

Hence, if p_0, q_0 denote the pressure and the radial velocity at any point due to the primary waves alone, and if p_1, q_1 denote the pressure and radial velocity due to the secondary waves alone, it is easily seen that the dissipation of energy within a distance R of the obstacle is given by

$$\int (p_0+p_1)(q_0+q_1)\,ds \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

where it has been assumed that q_0, q_1 are both measured inwards, and the integration is to be taken round the boundary of the surface $r = \mathrm{R}$.

Now the dissipation of energy due to the primary waves alone is given by $\int p_0 q_0\,ds$.

Hence it follows from (1) that the additional dissipation of energy due to the presence of the obstacle is given by

$$\int p_0 q_1\,ds + \int p_1 q_0\,ds + \int p_1 q_1\,ds \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

Now the rate at which energy is being carried across the surface $r = \mathrm{R}$ is $-\int p_1 q_1\,ds$, and hence from (2) it follows that the total additional dissipation of energy due to the presence of the obstacle is expressed by

$$\int p_0 q_1\,ds + \int p_1 q_0\,ds \, . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$$

where the integration is to be taken round the boundary of the surface $r = \mathrm{R}$.*

Since $\sigma\nu/c^2$ is in all cases a small fraction, it is clear from (6), § 3, that R may be great compared with the wave-length of the incident sound and yet such that $\sigma^2\nu\mathrm{R}/c^3$ is a small fraction. In this case we may neglect the imaginary part of $h\mathrm{R}$ in expressing the value of ϕ_0 and ϕ_1 at the surface $r = \mathrm{R}$. Also, if $\sigma\mathrm{R}/c$ is great, it follows that $|k\mathrm{R}|$ or $\lambda\mathrm{R}$ is very great, since their ratio is the small quantity $(\sigma\nu/c^2)^{1/2}$. Since, when $|k\mathrm{R}|$ is great, $\mathrm{D}_n(k\mathrm{R})$ approximates to the value

$$\left(\frac{2}{\pi k\mathrm{R}}\right)^{1/2} \iota^n e^{-\iota(\lambda\mathrm{R}/\sqrt{2}+1/4\pi)} e^{-\lambda\mathrm{R}/\sqrt{2}},$$

---

* This method of finding an expression for the loss of energy was kindly suggested to me by Prof. LAMB. I had previously obtained the same results by means of the dissipation function; but the work involved was very cumbrous.

we see that the ψ terms in the expression for the secondary waves are inappreciable at the external boundary owing to the exponential factor exp. $(-\lambda R/\sqrt{2})$.

We shall suppose then that R is great compared with the wave-length of the incident sound and yet such that $\sigma^2\nu R/c^3$ is small.

Let us now return to the consideration of (3). At the external boundary, $r = R$, we may write approximately

$$\phi_1 = \left(\frac{2}{\pi h R}\right)^{1/2} \sum_{n=0}^{\infty} \left[A_n \iota^n \cos n\vartheta e^{\iota(\sigma t - hR - {}^1/_4\pi)}\right],$$

where square brackets are used to denote that only the real part of the expression so enclosed is to be considered.

Hence we obtain

$$p_1 = -\rho_0 \frac{\partial \phi_1}{\partial t} = -\rho_0\sigma\left(\frac{2}{\pi h R}\right)^{1/2} \sum_{n=0}^{\infty} \left[A_n \iota^{n+1} \cos n\vartheta \,.\, e^{\iota(\sigma t - hR - {}^1/_4\pi)}\right],$$

$$q_1 = -\frac{\partial \phi_1}{\partial r} = h\left(\frac{2}{\pi h R}\right)^{1/2} \sum_{n=0}^{\infty} \left[A_n \iota^{n+1} \cos n\vartheta \,.\, e^{\iota(\sigma t - hR - {}^1/_4\pi)}\right].$$

Using these expressions for p_1 and q_1 we obtain

$$p_1 q_0 + p_0 q_1 = -\left(\frac{2}{\pi h R}\right)^{1/2} (\rho_0\sigma q_0 - h p_0) \sum_{n=0}^{\infty} \left[A_n \iota^{n+1} \cos n\vartheta e^{\iota(\sigma t - hR - {}^1/_4\pi)}\right]. \quad . \quad . \quad (4).$$

Again, since hR is great, we have from (1), § 3, approximately

$$\phi_0 = \left(\frac{2}{\pi h R}\right)^{1/2} \sin\left(hR + \tfrac{1}{4}\pi\right) \cos \sigma t + 2\sum_{n=1}^{\infty} \left(\frac{2}{\pi h R}\right)^{1/2} \sin\left(hR + \tfrac{1}{4}\pi - \tfrac{1}{2}n\pi\right) \cos\left(\sigma t + \tfrac{1}{2}n\pi\right) \cos n\vartheta.$$

Hence we obtain

$$p_0 = \rho_0\sigma\left(\frac{2}{\pi h R}\right)^{1/2} \sin\left(hR + \tfrac{1}{4}\pi\right) \sin \sigma t + 2\rho_0\sigma \sum_{\pi=1}^{\infty} \left(\frac{2}{\pi h R}\right)^{1/2} \sin\left(hR + \tfrac{1}{4}\pi - \tfrac{1}{2}n\pi\right)$$
$$\sin\left(\sigma t + \tfrac{1}{2}n\pi\right) \cos n\vartheta$$

and

$$q_0 = -h\left(\frac{2}{\pi h R}\right)^{1/2} \cos\left(hR + \tfrac{1}{4}\pi\right) \cos \sigma t - 2h \sum_{n=1}^{\infty} \left(\frac{2}{\pi h R}\right)^{1/2} \cos\left(hR + \tfrac{1}{4}\pi - \tfrac{1}{2}n\pi\right)$$
$$\cos\left(\sigma t + \tfrac{1}{2}n\pi\right) \cos n\vartheta.$$

Combining these expressions for p_0 and q_0 we find without difficulty

$$-(\rho_0\sigma q_0 - h p_0) = h\rho_0\sigma\left(\frac{2}{\pi h R}\right)^{1/2} \cos\left(\sigma t - hR - \tfrac{1}{4}\pi\right) + 2h\rho_0\sigma\left(\frac{2}{\pi h R}\right)^{1/2}$$
$$\sum_{n=1}^{\infty} (-)^n \cos\left(\sigma t - hR - \tfrac{1}{4}\pi\right) \cos n\vartheta.$$

Substituting in (4) and integrating with respect to ϑ we obtain

$$\int_0^{2\pi} (p_1 q_0 + p_0 q_1) R \, d\vartheta = 4\rho_0\sigma \sum_{n=0}^{\infty} \left[(-)^n A_n \iota^{n+1} \cos\left(\sigma t - hR - \tfrac{1}{4}\pi\right) e^{\iota(\sigma t - hR - {}^1/_4\pi)}\right]$$

of which the mean value is

$$2\rho_0\sigma \sum_{n=0}^{\infty} \left[(-)^n A_n \iota^{n+1}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

This last expression, then, represents the loss of energy to the primary waves in consequence of the presence of the obstacle.

From the value of A_n obtained in (12) of § 3 we see that the summation (5) consists of a series of terms arranged in descending order of magnitude. Consequently, in determining its value we shall limit our attention to the first two terms of the summation. Hence the total loss of energy to the primary waves is given very approximately by

$$2\rho_0\sigma\left[\iota A_0 + A_1\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6).$$

Let us first consider the case when λa is small. In this case we have, from (17) and (19) of § 3,

$$A_1 = -\frac{\sigma\nu}{c^2}\pi\left(\log\tfrac{1}{2}\lambda a + \gamma + \tfrac{1}{4}\iota\pi\right)^{-1},$$

$$A_0 = -\tfrac{1}{4}\pi h^2a^2\left\{1 + \tfrac{1}{2}h^2a^2\left(\log\tfrac{1}{2}ha + \gamma + \tfrac{1}{2}\iota\pi - \tfrac{3}{4}\right)\right\}.$$

Hence

$$\left[\iota A_0 + A_1\right] = \tfrac{1}{16}\pi^2h^4a^4 - \frac{\sigma\nu}{c^2}\pi\left(\log\tfrac{1}{2}\lambda a + \gamma\right)\left\{\left(\log\tfrac{1}{2}\lambda a + \gamma\right)^2 + \tfrac{1}{16}\pi^2\right\}^{-1}.$$

Now for small values of the radius a the first term of this last expression is small compared with the second, and consequently may be neglected. Hence the loss of energy to the primary waves is given approximately by

$$-2\rho_0\frac{\nu\sigma^2}{c^2}\pi\left(\log\tfrac{1}{2}\lambda a + \gamma\right)\left\{\left(\log\tfrac{1}{2}\lambda a + \gamma\right)^2 + \tfrac{1}{16}\pi^2\right\}^{-1} \quad . \quad . \quad . \quad . \quad . \quad (7),$$

in the case when λa is a small fraction.

The ratio of this last expression to $\rho_0\sigma^2a/c$, which represents the rate at which energy is incident in the primary waves upon the obstacle, is given by

$$-\frac{2\pi\nu}{ca}\left(\log\tfrac{1}{2}\lambda a + \gamma\right)\left\{\left(\log\tfrac{1}{2}\lambda a + \gamma\right)^2 + \tfrac{1}{16}\pi^2\right\}^{-1} \quad . \quad . \quad . \quad . \quad . \quad (8).$$

We may notice from this last result that, when λa is small, the proportion of the incident energy, which is lost to the primary waves, is very nearly proportional to the reciprocal of the radius, since the logarithmic terms will change more slowly as the radius changes. Hence, as in (7), the total energy lost to the primary waves is almost independent of the dimensions of the obstacle, provided these are small enough to satisfy the conditions under which the results (7) and (8) have been obtained, and provided also that we limit our attention to obstacles whose dimensions are of about the same order of magnitude. It might have been anticipated that the energy lost by friction in the neighbourhood of the obstacle would, in the case of very small obstacles, alter very slowly with the dimensions of the obstacle, and consequently that the effect upon the primary waves of very small obstacles would be almost independent of the dimensions of the obstacles, provided they were limited to be of the same order of magnitude.

Let us now turn to the case when λa is great. In this case we have approximately, from (17) and (19) of § 3,

$$A_1 = \tfrac{1}{2}\iota\pi h^2a^2[1+\sqrt{2}\,(\lambda a)^{-1}-\tfrac{1}{2}h^2a^2(\log\tfrac{1}{2}ha+\gamma+\tfrac{5}{4})-\iota\{\sqrt{2}\,(\lambda a)^{-1}+\tfrac{7}{4}(\lambda a)^{-2}+\tfrac{1}{4}\pi h^2a^2\}],$$
$$A_0 = -\tfrac{1}{4}\pi h^2a^2\{1+\tfrac{1}{2}h^2a^2(\log\tfrac{1}{2}ha+\gamma+\tfrac{1}{2}\iota\pi-\tfrac{3}{4})\}.$$

Hence we find approximately

$$\begin{aligned}[\iota A_0+A_1] &= \tfrac{1}{16}\pi^2h^4a^4+\tfrac{1}{8}\pi^2h^4a^4+\tfrac{1}{2}h^2a^2\{\sqrt{2}\,(\lambda a)^{-1}+\tfrac{7}{4}(\lambda a)^{-2}\},\\ &= \tfrac{3}{16}\pi^2h^4a^4+\tfrac{1}{2}\pi h^2a^2\{\sqrt{2}\,(\lambda a)^{-1}+\tfrac{7}{4}(\lambda a)^{-2}\}.\end{aligned}$$

Substituting this value for $[\iota A_0+A_1]$ in (6), we obtain, when λa is great, for the total loss of energy to the primary waves the formula

$$\tfrac{3}{8}\pi^2\rho_0\sigma h^4a^4+\pi\rho_0\sigma h^2a^2\{\sqrt{2}\,(\lambda a)^{-1}+\tfrac{7}{4}(\lambda a)^{-2}\}.$$

The ratio of this to $\rho_0\sigma^2a/c$, which represents the rate at which energy is incident upon the obstacle, is given by

$$\tfrac{3}{8}\pi^2\sigma^3a^3/c^3+\pi\sqrt{2}\,\sigma^{1/2}\nu^{1/2}/c+\tfrac{7}{4}\nu/(ca) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9),$$

which gives the proportion of the incident energy which is lost to the primary waves.

The first term in (9) is independent of the viscosity of the medium, and is obtained in the ordinary theory of a non-viscous air. The second and third terms of (9) represent the additional loss of energy to the primary waves consequent upon the viscosity of the medium. Further, since the ratio of the second to the third term of (9) is of order λa, it follows that the latter may be disregarded. Hence we see that, since a does not enter into the second term of (9), the additional proportional loss of energy consequent upon the viscosity is almost independent of the magnitude of the obstacles. In other words, the actual loss of energy in the primary waves due to friction is proportional to the radius of the obstructing cylinder if this be sufficiently large. This last result is clearly what might have been expected on physical grounds.

It remains to consider the case when λa is neither very small nor very great. In this case it is impossible to approximate to the values of the Bessel functions involved.

From (13) § 3 we have

$$A_1 = -\tfrac{1}{2}\iota\pi h^2a^2\frac{D_2(ka)}{D_0(ka)}.$$

Hence it follows that

$$[A_1] = \tfrac{1}{2}\pi h^2a^2.\tfrac{1}{2}\iota\left\{\frac{E_2(\lambda a e^{1/4\iota\pi})}{E_0(\lambda a e^{1/4\iota\pi})}-\frac{D_2(\lambda a e^{-1/4\iota\pi})}{D_0(\lambda a e^{-1/4\iota\pi})}\right\},$$

where

$$E_n(\zeta) = \frac{2}{\pi}\{(\log 2-\gamma+\tfrac{1}{2}\iota\pi)\,J_n(\zeta)-Y_n(\zeta)\},$$

and square brackets are, as before, used to denote that the real part only of the expression so enclosed is being considered.

Hence we have

$$[A_1] = -\tfrac{1}{4}\iota\pi h^2 a^2 (D_2E_0-D_0E_2)\,|D_0|^{-2}.$$

Now $[\iota A_0] = \frac{1}{16}\pi^2 h^4 a^4$, and consequently it may be neglected in comparison with $[A_1]$.

Hence the loss of energy to the primary waves is given by

$$\tfrac{1}{2}\iota\pi\rho_0\sigma h^2 a^2 (D_0E_2-D_2E_0)\,|D_0|^{-2}.$$

The ratio of this to $\rho_0\sigma^2 a/c$ is

$$\tfrac{1}{2}\iota\pi\frac{\sigma a}{c}(D_0E_2-D_2E_0)\,|D_0|^{-2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (10),$$

which therefore represents the proportion of the incident energy, which is lost to the primary waves.

On p. 253 will be found a table giving the ratio of the lost energy to that incident upon the cylindrical obstacle in a number of different cases. For wires and cylindrical rods of comparatively large radius it is necessary to use the formula (9); the results for wires of radii 10 cm., 1 cm., and ·1 cm. have been deduced from this formula. The formula (8) is applicable when the radius of the obstacle is very small, and the results for wires of radius 10^{-3} cm. have been obtained from it. When the radius of the wire is of order 10^{-2} cm., neither of these approximate formulæ is applicable, and it becomes necessary to calculate the results directly from (10); the value of the ratio of the lost energy to that incident upon the wire has been worked out in this case for only a few values of the wave-length on account of the laborious nature of the work involved.

It should be added that in the table given on p. 253 λ denotes the wave-length of the incident sound measured in centimetres, and K denotes the ratio of the lost energy to that incident upon the obstacle.

I have also worked out the values of K for wires of different diameters in the case when the wave-length of the incident sound is 250 cm. The results are arranged below :—

λ = 250 cm.						
a	1 cm.	·1 cm.	·05 cm.	·01 cm.	·005 cm.	·001 cm.
K	$\cdot 16\ 10^{-2}$	$\cdot 15\ 10^{-2}$	$\cdot 21\ 10^{-2}$	$\cdot 26\ 10^{-2}$	$\cdot 32\ 10^{-2}$	$\cdot 86\ 10^{-2}$

a = 10 cm.		a = 1 cm.		a = ·1 cm.		a = ·01 cm.		a = ·001 cm.	
λ.	K.	λ.	K.	λ.	K.	λ.	K.	λ.	K.
300	35·3 10^{-3}	30	38·1 10^{-3}	10	7·9 10^{-3}	10	1·07 10^{-2}	39·5	1·35 10^{-2}
400	15·5 10^{-3}	40	17·9 10^{-3}	20	5·1 10^{-3}	39·5	·75 10^{-2}	50	1·15 10^{-2}
500	8·3 10^{-3}	50	10·5 10^{-3}	30	4·1 10^{-3}	158	·29 10^{-2}	100	1·02 10^{-2}
600	5·2 10^{-3}	60	7·1 10^{-3}	40	3·5 10^{-3}	439	·21 10^{-2}	200	·89 10^{-2}
700	3·5 10^{-3}	70	5·3 10^{-3}	50	3·2 10^{-3}	987	·16 10^{-2}	300	·84 10^{-2}
800	2·6 10^{-3}	80	4·2 10^{-3}	60	2·9 10^{-3}	3950	·14 10^{-2}	400	·81 10^{-2}
900	2·0 10^{-3}	90	3·6 10^{-3}	70	2·7 10^{-3}			500	·78 10^{-2}
1000	1·6 10^{-3}	100	3·3 10^{-3}	80	2·5 10^{-3}			600	·76 10^{-2}
		200	1·8 10^{-3}	90	2·3 10^{-3}			700	·74 10^{-2}
		300	1·3 10^{-3}	100	2·2 10^{-3}			800	·73 10^{-2}
		400	1·1 10^{-3}	200	1·6 10^{-3}			900	·72 10^{-2}
		500	1·0 10^{-3}	300	1·3 10^{-3}			1000	·71 10^{-2}
		600	·9 10^{-3}	400	1·1 10^{-3}				
		700	·8 10^{-3}	500	1·0 10^{-3}				
		800	·8 10^{-3}	600	·9 10^{-3}				
		900	·7. 10^{-3}	700	·8 10^{-3}				
		1000	·7 10^{-3}	800	·8 10^{-3}				
				900	·7 10^{-3}				
				1000	·7 10^{-3}				

§ 5. *Extension to the Problem of a Number of Cylindrical Obstacles.*—Let us suppose that there are n parallel wires per unit area of a plane perpendicular to their common direction. At a distance r from the axis of any one of these wires, great compared with the wave-length of the incident sound, the secondary waves due to that wire will be given very approximately in all cases by

$$\phi_1 = \mathrm{A}_0 \, \mathrm{D}_0 (hr) + \mathrm{A}_1 \, \mathrm{D}_1 (hr) \cos \vartheta.$$

Since hr is great, we may write for all values of n

$$\mathrm{D}_n (hr) = \left(\frac{2}{\pi hr}\right)^{1/2} \iota^n e^{-\iota (hr + 1/4\pi)},$$

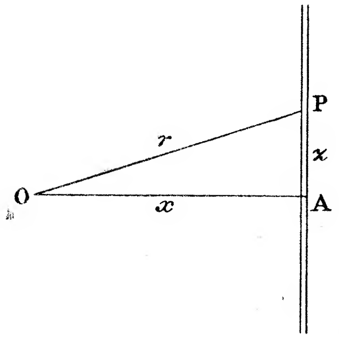
and ϕ_1 takes the form

$$\phi_1 = (\mathrm{A}_0 + \mathrm{A}_1 \iota \cos \vartheta) \left(\frac{2}{\pi hr}\right)^{1/2} e^{-\iota (hr + 1/4\pi)}.$$

Along the course of the primary waves ($\vartheta = \pi$) this reduces to

$$\phi_1 = (\mathrm{A}_0 - \mathrm{A}_1 \iota) \left(\frac{2}{\pi hr}\right)^{1/2} e^{-\iota (hr + 1/4\pi)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

Consider now the wires which occupy a thin stratum dx perpendicular to the course of the primary waves. Let AP be the section of this stratum by a plane at right angles to the wires and let O be the point, at which the vibration is to be estimated at a great distance from the stratum.



If AP $= z$, the element of area is $dx \, . \, dz$, and consequently the number of wires cutting it is $n \, . \, dx \, . \, dz$.

Also, if OP $= r$, AO $= -x$, then $r^2 = x^2 + z^2$ and $z \, dz = r \, dr$.

The resultant at O of all the secondary vibrations, which issue from the stratum, is by (1)

$$2n \, dx \int_{-x}^{\infty} (\mathrm{A}_0 - \mathrm{A}_1 \iota) \left(\frac{2}{\pi h}\right)^{1/2} e^{-\iota (hr + 1/4\pi)} \frac{r^{1/2} \, dr}{\sqrt{(r^2 - x^2)}}.$$

Writing $r = -x + \eta$, we obtain

$$2n \, dx \left(\frac{2}{\pi h}\right)^{1/2} (\mathrm{A}_0 - \mathrm{A}_1 \iota) \, e^{\iota (hx - 1/4\pi)} \int_0^{\infty} \frac{e^{-\iota h \eta} (-x + \eta)^{1/2}}{\eta^{1/2} (-2x + \eta)^{1/2}} \, d\eta.$$

In evaluating this last expression we may assume η / x to be very small; it then takes the form

$$2n \, dx \left(\frac{2}{\pi h}\right)^{1/2} (\mathrm{A}_0 - \mathrm{A}_1 \iota) \, e^{\iota (hx - 1/4\pi)} \int_0^{\infty} (2\eta)^{-1/2} e^{-\iota h \eta} \, d\eta \quad . \quad . \quad . \quad . \quad . \quad (2).$$

Now

$$\int_0^{\infty} \eta^{-1/2} e^{-\iota h \eta} \, d\eta = 2 \int_0^{\infty} e^{-\iota h v^2} \, dv.$$

Also by a well known result we have

$$\int_0^\infty \cos(hv^2)\,dv = \int_0^\infty \sin(hv^2)\,dv = \left(\frac{\pi}{8h}\right)^{1/2}.$$

Consequently

$$\int_0^\infty \eta^{-1/2}e^{-\iota h\eta}\,d\eta = \sqrt{2}\,.\,(1-\iota)\left(\frac{\pi}{4h}\right)^{1/2}.$$

Substituting this result in (2) and restoring the time factor, we obtain for the resultant at O of all the secondary vibrations coming from the stratum dx

$$-2n\,.\,dx\,.\,(\mathrm{A}_1+\iota\mathrm{A}_0)\,h^{-1}e^{\iota(hx+\sigma t)}\quad .\ .\ .\ .\ .\ .\ .\ .\ .\quad (3).$$

When λa is small, A_1 is great compared with A_0; neglecting A_0 and using the expression for A_1 given in (15), § 3, we obtain instead of (3)

$$2n\,.\,dx\,.\,\frac{\nu\pi}{c}\left(\log\tfrac{1}{2}\lambda a+\gamma+\tfrac{1}{4}\iota\pi\right)^{-1}e^{\iota(hx+\sigma t)},$$

of which the real part is

$$2n\,dx\,\frac{\nu\pi}{c}\left\{(\log\tfrac{1}{2}\lambda a+\gamma)\cos(hx+\sigma t)+\tfrac{1}{4}\pi\sin(hx+\sigma t)\right\}\Big/\left\{(\log\tfrac{1}{2}\lambda a+\gamma)^2+\tfrac{1}{16}\pi^2\right\}\quad (4).$$

To this is to be added the corresponding expression for the primary wave

$$\phi_0 = \cos(hx+\sigma t).$$

The coefficient of $\cos(hx+\sigma t)$ is thus altered by the obstacles in the layer dx from unity to

$$1+2n\,dx\,\frac{\nu\pi}{c}(\log\tfrac{1}{2}\lambda a+\gamma)\Big/\left\{(\log\tfrac{1}{2}\lambda a+\gamma)^2+\tfrac{1}{16}\pi^2\right\}.$$

Thus, if E be the energy in the incident waves, we have

$$d\mathrm{E}/\mathrm{E} = 4n.dx\frac{\nu\pi}{c}(\log\tfrac{1}{2}\lambda a+\gamma)\Big/\left\{(\log\tfrac{1}{2}\lambda a+\gamma)^2+\tfrac{1}{16}\pi^2\right\}.$$

Integrating this, we obtain

$$\mathrm{E} = \mathrm{E}_0e^{-\alpha x},$$

where E_0 is the energy in the primary waves at incidence and α is given by

$$\alpha = -4\frac{n\nu\pi}{c}(\log\tfrac{1}{2}\lambda a+\gamma)\Big/\left\{(\log\tfrac{1}{2}\lambda a+\gamma)^2+\tfrac{1}{16}\pi^2\right\}\quad .\ .\ .\ .\ .\quad (5).$$

The coefficient of $\sin(hx+\sigma t)$ in (4) gives the refractivity of the medium as modified by the wires. If δ be the retardation due to the wires of the stratum dx,

$$\delta = \tfrac{1}{2}\frac{n\nu\pi^2}{\sigma}dx\Big/\left\{(\log\tfrac{1}{2}\lambda a+\gamma)^2+\tfrac{1}{16}\pi^2\right\}.$$

Hence, if μ be the refractive index as modified by the wires

$$\mu-1 = \tfrac{1}{2}n\pi^2\frac{\nu}{\sigma}\Big/\{(\log\tfrac{1}{2}\lambda a+\gamma)^2+\tfrac{1}{16}\pi^2\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6).$$

Hence we have

$$\mu-1 = \tfrac{1}{2}\pi\frac{\nu}{\sigma a^2}p\Big/\{(\log\tfrac{1}{2}\lambda a+\gamma)^2+\tfrac{1}{16}\pi^2\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

where p denotes the ratio, *assumed small*, of the volume occupied by the wires to the total volume.

Let us now consider the case when λa is great. In this case we have, from (17) and (18) § 3

$$\mathrm{A}_1+\iota\mathrm{A}_0 = \tfrac{1}{4}\iota\pi h^2a^2[1+2\sqrt{2}\,(\lambda a)^{-1}-\tfrac{3}{2}h^2a^2(\log\tfrac{1}{2}ha+\gamma+\tfrac{7}{12})$$
$$-\iota\{2\sqrt{2}\,(\lambda a)^{-1}+\tfrac{7}{2}(\lambda a)^{-2}+\tfrac{3}{4}\pi h^2a^2\}].$$

Substituting this expression for $\mathrm{A}_1+\iota\mathrm{A}_0$ in (3), we obtain for the resultant at 0 of all the secondary vibrations coming from the stratum dx

$$\phi_1 = -\tfrac{1}{2}\iota n\pi\,dx\,.\,ha^2[1+2\sqrt{2}\,(\lambda a)^{-1}-\tfrac{3}{2}h^2a^2(\log\tfrac{1}{2}ha+\gamma+\tfrac{7}{12})$$
$$-\iota\{2\sqrt{2}\,(\lambda a)^{-1}+\tfrac{7}{2}(\lambda a)^{-2}+\tfrac{3}{4}\pi h^2a^2\}]\,e^{\iota(hx+\sigma t)}$$

of which the real part is

$$-\tfrac{1}{2}n\pi.dx.ha^2\{2\sqrt{2}\,(\lambda a)^{-1}+\tfrac{7}{2}(\lambda a)^{-2}+\tfrac{3}{4}\pi h^2a^2\}\cos(hx+\sigma t)$$
$$+\tfrac{1}{2}n\pi.dx.ha^2\{1+2\sqrt{2}\,(\lambda a)^{-1}-\tfrac{3}{2}h^2a^2(\log\tfrac{1}{2}ha+\gamma+\tfrac{7}{12})\}\sin(hx+\sigma t) \quad . \quad (8).$$

To this is to be added the expression for the primary waves

$$\phi_0 = \cos(hx+\sigma t).$$

The coefficient of $\cos(hx+\sigma t)$ is thus altered by the obstacles in the layer dx from unity to

$$1-\tfrac{1}{2}n\pi ha^2\{2\sqrt{2}\,(\lambda a)^{-1}+\tfrac{7}{2}(\lambda a)^{-2}+\tfrac{3}{4}\pi h^2a^2\}\,dx.$$

Hence, if E be the energy in the incident waves, we have

$$d\mathrm{E}/\mathrm{E} = -n\pi ha^2\{2\sqrt{2}\,(\lambda a)^{-1}+\tfrac{7}{2}(\lambda a)^{-2}+\tfrac{3}{4}\pi h^2a^2\}\,dx.$$

Integrating this, we obtain

$$\mathrm{E} = \mathrm{E}_0e^{-\alpha x},$$

where

$$\alpha = n\pi ha^2\{2\sqrt{2}\,(\lambda a)^{-1}+\tfrac{7}{2}(\lambda a)^{-2}+\tfrac{3}{4}\pi h^2a^2\}.$$

When σ/c and $(\sigma/\nu)^{1/2}$ are substituted for h and λ respectively, this takes the form

$$\alpha = 2\sqrt{2}n\pi.a.\sigma^{1/2}\nu^{1/2}/c+\tfrac{7}{2}n\pi\nu/c+\tfrac{3}{4}n\pi^2\sigma^3a^4/c^4 \quad . \quad . \quad . \quad . \quad . \quad (9).$$

The second term of (8) gives the refractivity of the medium as modified by the wires. If δ be the retardation due to the wires of the stratum dx

$$\delta = \tfrac{1}{2}n\pi a^2\left\{1+2\sqrt{2}\,(\lambda a)^{-1}-\tfrac{3}{2}\frac{\sigma^2 a^2}{c^2}(\log\tfrac{1}{2}\sigma a/c+\gamma+\tfrac{7}{12})\right\}.$$

Hence, if μ be the refractive index as modified by the wires, we have approximately

$$\mu-1 = \tfrac{1}{2}\{1+2\sqrt{2}\,(\lambda a)^{-1}\}\,p \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

where p denotes the ratio, assumed small as before, of the volume occupied by the wires to the total volume.

If the waves of sound traverse a medium in which a number of parallel wires are arranged, then the reciprocal of α will determine the distance which the waves will travel before the intensity of the sound is diminished in the ratio of $1/e$. For sound waves of wave-length 10 cm. passing through a medium, in which there are 100 parallel wires of radius 10^{-2} cm. per unit area of a section perpendicular to the wires, this distance is 47 cm. For greater wave-lengths the distance is greater. It seems hardly probable that any arrangement of wires could improve the acoustic properties of a room unless some other factor than viscosity is taken into account. Of course, if n is made sufficiently great, the reciprocal of α may become very small; but it seems probable that it would be difficult to arrange the wires so closely that n should be greater than 10^3. If it was possible to arrange wires of radius 10^{-3} cm. so closely that n was 10^4, then the intensity of sound of wave-length 40 cm. would be diminished in the ratio $1/e$ after passing through a thickness of less than 4 cm. Such a contrivance could hardly, however, be carried out in practice.

§ 6. *Problems Relating to Spherical Obstacles.*—We require first a solution of the equations of motion suitable to such problems. Differentiating the equations of vibration (1), § 1, with regard to x, y, z respectively and adding, we obtain with the help of (2), § 2,

$$\frac{\partial^2 s}{\partial t^2} = c^2\nabla^2 s+\tfrac{4}{3}\nu\nabla^2\frac{\partial s}{\partial t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

If we now assume a time factor $e^{\iota\sigma t}$, this equation takes the form

$$(\nabla^2+h^2)\,s = 0, \quad \text{where} \quad h^2 = \sigma^2/(c^2+\tfrac{4}{3}\iota\nu\sigma) \quad . \quad . \quad . \quad . \quad (2), (3).$$

Also the equations of motion (1), § 1, may with the help of (2) and (3), § 2, be written in the form

$$(\nabla^2+k^2)\,u = (k^2-h^2)\frac{\partial\phi}{\partial x}, \quad (\nabla^2+k^2)\,v = (k^2-h^2)\frac{\partial\phi}{\partial y}, \quad (\nabla^2+k^2)\,w = (k^2-h^2)\frac{\partial\phi}{\partial z} \quad . \quad (4),$$

where

$$\phi = \iota\sigma h^{-2}s = -h^{-2}\,\mathrm{div}\,(u, v, w) \quad \text{and} \quad k^2 = -\iota\sigma/\nu \quad . \quad . \quad . \quad (5), (6).$$

These equations (4) are satisfied by

$$(u, v, w) = \mathrm{grad}\,\phi,$$

where ϕ is any solution of the equation

$$(\nabla^2+h^2)\,\phi = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7).$$

The complete solution of the equations of motion will be given by

$$(u,\ v,\ w) = (u',\ v',\ w') + \operatorname{grad}\phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8),$$

where u', v', w' satisfy the equations

$$\left.\begin{array}{l} (\nabla^2+k^2)\,u' = 0, \qquad (\nabla^2+k^2)\,v' = 0, \qquad (\nabla^2+k^2)\,w' = 0 \\ \text{together with} \qquad \dfrac{\partial u'}{\partial x}+\dfrac{\partial v'}{\partial y}+\dfrac{\partial w'}{\partial z} = 0 \end{array}\right\} \quad . \quad . \quad . \quad . \quad (9).$$

The solution of these equations suitable to the case of waves diverging from the origin is given by

$$u' = \Sigma\left\{(n+1)\,f_{n-1}(kr)\frac{\partial\omega_n}{\partial x} - nk^2r^{2n+3}f_{n+1}(kr)\frac{\partial}{\partial x}\left(\frac{\omega_n}{r^{2n+1}}\right)\right\} + \Sigma f_n(kr)\left(y\frac{\partial}{\partial z}-z\frac{\partial}{\partial y}\right)\chi_n \quad . \quad (10),$$

with corresponding expressions for v' and w'. Here ω_n and χ_n are solid harmonics of positive degree n, and $f_n(kr)$ is a function of kr given by the relation

$$f_n(\zeta) = \left(-\frac{d}{\zeta\,d\zeta}\right)^n\frac{e^{-\iota\zeta}}{\zeta} = \Psi_n(\zeta)-\iota\psi(\zeta) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11),$$

where $\Psi_n(\zeta)$ and $\psi_n(\zeta)$ satisfy the relations

$$\Psi_n(\zeta) = \left(-\frac{d}{\zeta\,d\zeta}\right)^n\frac{\cos\zeta}{\zeta}, \qquad \psi_n(\zeta) = \left(-\frac{d}{\zeta\,d\zeta}\right)^n\frac{\sin\zeta}{\zeta} \quad . \quad . \quad . \quad . \quad (12).$$

The general formulæ for $\psi_n(\zeta)$, $\Psi_n(\zeta)$, and $f_n(\zeta)$, are

$$\psi_n(\zeta) = \frac{1}{1.3.5\ldots(2n+1)}\left\{1-\frac{\zeta^2}{2\,(2n+3)}+\frac{\zeta^4}{2.4.(2n+3)\,(2n+5)}-\ldots\right\} \quad . \quad . \quad . \quad (13),$$

$$\Psi_n(\zeta) = \frac{1.3.5\ldots(2n-1)}{\zeta^{2n+1}}\left\{1-\frac{\zeta^2}{2\,(1-2n)}+\frac{\zeta^4}{2.4.(1-2n)\,(3-2n)}-\ldots\right\} \quad . \quad . \quad . \quad (14),$$

$$f_n(\zeta) = \frac{\iota^n e^{-\iota\zeta}}{\zeta^{n+1}}\left\{1+\frac{n.(n+1)}{2\iota\zeta}+\frac{(n-1)\,n\,(n+1)\,(n+2)}{2.4.(\iota\zeta)^2}+\ldots\frac{1.2.3\ldots2n}{2.4.6\ldots2n\,(\iota\zeta)^n}\right\} \quad . \quad (15).$$

The functions $\psi_n(\zeta)$, $\Psi_n(\zeta)$, and $f_n(\zeta)$, all satisfy recurrence formulæ of the types

$$\psi'_n(\zeta) = -\zeta\psi_{n+1}(\zeta), \qquad \zeta\psi'_n(\zeta) + (2n+1)\,\psi_n(\zeta) = \psi_{n-1}(\zeta) \quad . \quad (16), (17);$$

these will be found useful hereafter in reductions.

Returning now to the consideration of the equations of vibration, we find from (8) and (10) that the general solution suitable to divergent waves is given by

$$u = \frac{\partial \phi}{\partial x} + \sum_{n=0}^{\infty} \left\{ (n+1) f_{n-1}(kr) \frac{\partial \omega_n}{\partial x} - nk^2 r^{2n+3} f_{n+1}(kr) \frac{\partial}{\partial x}\left(\frac{\omega_n}{r^{2n+1}}\right) \right\}$$
$$+ \sum_{n=0}^{\infty} f_n(kr) \left(y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y} \right) \chi_n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18),$$

with corresponding expressions for v and w; ϕ here represents the general solution of (7) suitable to divergent waves.

Hence

$$\phi = \sum_{n=0}^{\infty} f_n(hr) . \phi_n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19),$$

where ϕ_n is a solid harmonic of positive degree n.

If the motion is in planes through the axis of x, and is symmetrical about that axis, the solution takes the form

$$u = \frac{\partial \phi}{\partial x} + \sum_{n=0}^{\infty} \left\{ (n+1) f_{n-1}(kr) \frac{\partial \omega_n}{\partial x} - nk^2 r^{2n+3} f_{n+1}(kr) \frac{\partial}{\partial x}\left(\frac{\omega_n}{r^{2n+1}}\right) \right\} \quad . \quad . \quad (20),$$

with corresponding expressions for v and w. Further we have

$$\phi = \sum_{n=0}^{\infty} A_n f_n(hr)\, r^n P_n(\mu), \qquad \omega_n = B_n r^n P_n(\mu) \quad . \quad . \quad . \quad (21), (22),$$

where A_n and B_n are arbitrary constants.

We write as before

$$k = \lambda e^{-\frac{1}{4}\iota\pi}, \qquad \text{where} \qquad \lambda = (\sigma/\nu)^{1/2} \quad . \quad . \quad . \quad . \quad (23), (24).$$

From (15) it is seen that $f_n(kr)$ contains the exponential factor $e^{-(\frac{1}{2}\sigma/\nu)^{\frac{1}{2}} r}$ or $e^{-\frac{1}{2}\sqrt{2}\lambda r}$; consequently since λr becomes very great within a short distance of the origin, it is clear that, at a moderate distance from the origin, those parts of the expression (20), which depend on the functions $f_n(kr)$, become inappreciable and may be neglected. Hence, at a sufficient distance from the origin, we may write

$$(u, v, w) = \text{grad}\, \phi, \qquad \text{where} \qquad \phi = \sum_{n=0}^{\infty} A_n f_n(hr)\, r^n P_n(\mu).$$

§ 7. *The Incidence of Plane Waves of Sound upon an Obstructing Sphere.*—We may now consider the effect of a spherical obstacle upon a train of plane waves of sound. Suppose the centre of the obstructing sphere to be at the origin, and the sound to be propagated in the negative direction along the axis of x; then, as before, we may assume for the incident waves

$$(u_0, v_0, w_0) = \text{grad}\, \phi_0, \quad \text{where} \quad \phi_0 = e^{\iota hx} \quad . \quad . \quad . \quad . \quad . \quad (1), (2).$$

Expanding in terms of the functions ψ_n, we obtain

$$\phi_0 = \sum_{n=0}^{\infty} (2n+1)\, \iota^n h^n \psi_n(hr)\, r^n \mathrm{P}_n(\mu) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

The scattered waves will be symmetrical about the axis of x, and so we may assume for them

$$(u_1, v_1, w_1) = \operatorname{grad}\phi_1 + \sum_{n=0}^{\infty} \left\{ (n+1) f_{n-1}(kr) \operatorname{grad}\omega_n - nk^2 r^{2n-3} f_{n+1}(kr) \operatorname{grad}\left(\frac{\omega_n}{r^{2n+1}}\right) \right\} \quad . \quad (4),$$

where

$$\phi_1 = \sum_{n=0}^{\infty} \mathrm{A}_n f_n(hr)\, r^n \mathrm{P}_n(\mu), \qquad \omega_n = \mathrm{B}_n r^n \mathrm{P}_n(\mu) \quad . \quad . \quad . \quad . \quad (5), (6).$$

Now, by means of the recurrence formulæ, it may be proved without difficulty that

$$\frac{\partial}{\partial x}\{f_n(hr)\, r^n \mathrm{P}_n\} = \frac{1}{2n+1}\left\{ f_{n-1}(hr) \frac{\partial}{\partial x}(r^n \mathrm{P}_n) + h^2 r^{2n+3} f_{n+1}(hr) \frac{\partial}{\partial x}\left(\frac{\mathrm{P}_n}{r^{n+1}}\right) \right\} \quad . \quad (7).$$

At the surface of the spherical obstacle $r = a$, we must have

$$u_0 + u_1 = 0, \qquad v_0 + v_1 = 0, \qquad w_0 + w_1 = 0.$$

Hence, when $r = a$, we have

$$\operatorname{grad}(\phi_0 + \phi_1) + \sum_{n=0}^{\infty} \left\{ (n+1) f_{n-1}(kr) \operatorname{grad}\omega_n - nk^2 r^{2n+3} f_{n+1}(kr) \operatorname{grad}\left(\frac{\omega_n}{r^{2n+1}}\right) \right\} = 0.$$

Introducing the expressions given above for ϕ_0, ϕ_1, ω_n, we obtain for all values of n, when $r = a$,

$$\begin{aligned} &\operatorname{grad}\{(2n+1)\, \iota^n h^n \psi_n(hr)\, r^n \mathrm{P}_n\} + \operatorname{grad}\{\mathrm{A}_n f_n(hr)\, r^n \mathrm{P}_n\} \\ &\qquad + (n+1) f_{n-1}(kr)\, \mathrm{B}_n \operatorname{grad}(r^n \mathrm{P}_n) - nk^2 r^{2n+3} f_{n+1}(kr)\, \mathrm{B}_n \operatorname{grad}\left(\frac{\mathrm{P}_n}{r^{n+1}}\right) = 0. \end{aligned}$$

Hence, by means of the identity (7), we find that the following expression

$$\begin{aligned} &\iota^n h^n \left[\psi_{n-1}(ha) \frac{\partial}{\partial x}(r^n \mathrm{P}_n) + h^2 a^{2n+3} \psi_{n+1}(ha) \frac{\partial}{\partial x}\left(\frac{\mathrm{P}_n}{r^{n+1}}\right) \right] \\ &\qquad + \frac{1}{2n+1} \mathrm{A}_n \left[f_{n-1}(ha) \frac{\partial}{\partial x}(r^n \mathrm{P}_n) + h^2 a^{2n+3} f_{n+1}(ha) \frac{\partial}{\partial x}\left(\frac{\mathrm{P}_n}{r^{n+1}}\right) \right] \\ &\qquad + \mathrm{B}_n \left[(n+1) f_{n-1}(ka) \frac{\partial}{\partial x}(r^n \mathrm{P}_n) - nk^2 a^{2n+3} f_{n+1}(ka) \frac{\partial}{\partial x}\left(\frac{\mathrm{P}_n}{r^{n+1}}\right) \right], \end{aligned}$$

and two other similar expressions must vanish, when $r = a$, for all values of n.

These three conditions will be satisfied if

$$\mathrm{A}_0 f_1(ha) = -\psi_1(ha) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8),$$

and if for all values of $n > 0$,

$$\left.\begin{aligned}\frac{1}{2n+1}\mathrm{A}_n f_{n-1}(ha)+\mathrm{B}_n(n+1)f_{n-1}(ka) &= -\iota^n h^n \psi_{n-1}(ha)\\ \frac{1}{2n+1}\mathrm{A}_n h^2a^2 f_{n+1}(ha)-\mathrm{B}_n nk^2a^2 f_{n+1}(ka) &= -\iota^n h^n h^2a^2\psi_{n+1}(ha)\end{aligned}\right\} \quad . \quad . \quad . \quad (9).$$

These equations (8) and (9) are sufficient to determine the various constants in the expression (4) for the scattered sound. In the process of approximating to the values of these constants we shall limit ourselves, as before, to the case when ha is a small fraction; in other words, we shall suppose that the radius of the obstacle is small compared with the wave-length of the incident sound. We shall also suppose that the gas in question is the air of the atmosphere. This will make $\sigma\nu/c^2$ a small fraction for all wave-lengths within the limits of audibility.

With this assumption we may write, as before,

$$h = \frac{\sigma}{c}(1-\tfrac{2}{3}\iota\nu\sigma/c^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10).$$

Eliminating A_n between the equations (9), we obtain

$$\begin{aligned}&\mathrm{B}_n\{(n+1)h^2a^2f_{n+1}(ha)f_{n-1}(ka)+nf_{n-1}(ha)k^2a^2f_{n+1}(ka)\}\\ &\qquad = -\iota^n h^n . h^2a^2\{\psi_{n-1}(ha)f_{n+1}(ha)-\psi_{n+1}(ha)f_{n-1}(ha)\} \quad . \quad (11).\end{aligned}$$

Now with the help of a well known result in the theory of Bessel functions it may be proved that

$$\psi_{n-1}(ha)f_{n+1}(ha)-\psi_{n+1}(ha)f_{n-1}(ha) = (2n+1)h^{-(2n+3)}a^{-(2n+3)}.$$

Hence equation (11) takes the form

$$\mathrm{B}_n\{(n+1)h^2a^2f_{n+1}(ha)f_{n-1}(ka)+nf_{n-1}(ha)k^2a^2f_{n+1}(ka)\} = -(2n+1)\iota^n h^n . h^{-(2n+1)} . a^{-(2n+1)} \quad (12).$$

Retaining only the principal term in the coefficient of B_n, we may write approximately

$$\mathrm{B}_n(n+1)\frac{(2n+2)!}{2^{n+1}(n+1)!}f_{n-1}(ka) = -\iota^n h^n(2n+1),$$

whence we have

$$\mathrm{B}_n = -\iota^n h^n\frac{2^n . n!}{(n+1)(2n)!}\cdot\frac{1}{f_{n-1}(ka)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13),$$

to the same degree of approximation.

Next eliminating B_n between the equations (9), we obtain

$$\begin{aligned}&\frac{1}{2n+1}\mathrm{A}_n[(n+1)h^2a^2f_{n+1}(ha)f_{n-1}(ka)+nf_{n-1}(ha)k^2a^2f_{n+1}(ka)]\\ &\qquad = -\iota^n h^n[n\psi_{n-1}(ha)k^2a^2f_{n+1}(ka)+(n+1)h^2a^2\psi_{n+1}(ha)f_{n-1}(ka)] \quad . \quad (14).\end{aligned}$$

Hence by use of the expressions (13) and (15) § 6 for $\psi_n(\zeta)$ and $f_n(\zeta)$, we have approximately, when ha is small, for all values of $n > 0$,

$$\mathrm{A}_n = -\iota^n h^n \frac{n}{n+1} \frac{h^{2n+1} \,.\, a^{2n+1}}{1^2 \,.\, 3^2 \ldots (2n-1)^2} k^2 a^2 f_{n+1}(ka) / f_{n-1}(ka) \quad . \quad . \quad . \quad (15).$$

We shall find it necessary to obtain a closer approximation to the value of A_1. Writing $n = 1$ in (14) we have

$$\tfrac{1}{3}\mathrm{A}_1\{2h^2a^2 f_2(ha) f_0(ka) + f_0(ha) k^2a^2 f_2(ka)\} = -\iota h\{\psi_0(ha) k^2a^2 f_2(ka) + 2h^2a^2 \psi_2(ha) f_0(ka)\}. \ (16).$$

Now from (15) § 6 we have approximately, since ha is small,

$$f_0(ha) = h^{-1}a^{-1}(1 - \iota ha - \tfrac{1}{2}h^2a^2), \qquad 2h^2a^2 f_2(ha) = 6h^{-3}a^{-3}(1 + \tfrac{1}{6}h^2a^2 + \tfrac{1}{24}h^4a^4).$$

Substituting these expressions in (16) and making use of (13) § 6, we obtain

$$\begin{aligned} 2\mathrm{A}_1 h^{-3}a^{-3} = \iota h\,[1 - 3\iota k^{-1}a^{-1} - 3k^{-2}a^{-2} - h^2a^2(\tfrac{3}{10} + \tfrac{3}{2}k^{-2}a^{-2} - 3\iota k^{-3}a^{-3} - \tfrac{3}{2}k^{-4}a^{-4}) \\ - \tfrac{1}{2}h^3a^3(\tfrac{1}{3}\iota + 2k^{-1}a^{-1} - 5\iota k^{-2}a^{-2} - 6k^{-3}a^{-3} + 3\iota k^{-4}a^{-4})]. \end{aligned}$$

Writing $\lambda a e^{-1/_4 \iota\pi}$ for ka we obtain finally

$$\begin{aligned} \mathrm{A}_1 = \tfrac{1}{2}h^4a^3\,[\tfrac{3}{2}\sqrt{2}\lambda^{-1}a^{-1} + 3\lambda^{-2}a^{-2} + \tfrac{1}{2}h^2a^2(3\lambda^{-2}a^{-2} + 3\sqrt{2}\lambda^{-3}a^{-3}) \\ - \tfrac{1}{2}h^3a^3(\tfrac{1}{3} + \sqrt{2}\lambda^{-1}a^{-1} + 3\sqrt{2}\lambda^{-3}a^{-3} - 3\lambda^{-4}a^{-4})] \\ + \iota\,\tfrac{1}{2}h^4a^3\,[1 + \tfrac{3}{2}\sqrt{2}\lambda^{-1}a^{-1} - \tfrac{1}{2}h^2a^2(\tfrac{3}{5} + 3\sqrt{2}\lambda^{-3}a^{-3} + 3\lambda^{-4}a^{-4}) \\ - \tfrac{1}{2}h^3a^3(\sqrt{2}\lambda^{-1}a^{-1} + 5\lambda^{-2}a^{-2} - 3\sqrt{2}\lambda^{-3}a^{-3})] \quad . \quad . \quad . \quad (17). \end{aligned}$$

By a similar process we obtain from (8),

$$\mathrm{A}_0 = -\tfrac{1}{3}h^3a^3(1 - \tfrac{3}{5}h^2a^2 + \tfrac{1}{3}\iota h^3a^3) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18).$$

§ 8. Having obtained approximate values for the various constants involved in the expressions for the secondary waves, we may now proceed to estimate the additional loss of energy consequent upon the presence of the obstacle. The method adopted is exactly similar to that of which we made use in the case of cylindrical obstacles. As above, it is easily seen that the total additional dissipation of energy due to the presence of the obstacle is given by

$$\iint p_0 q_1 \, d\mathrm{S} + \iint p_1 q_0 \, d\mathrm{S} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where the suffixed letters have the same meaning as in § 4, and the integration is to be taken over the surface of a sphere of radius R concentric with the obstacle. As before, we shall suppose that R is great compared with the wave-length of the incident sound, and yet such that $\sigma^2\nu\mathrm{R}/c^3$ is a small fraction. By this assumption we are enabled to neglect the imaginary part of $h\mathrm{R}$ and also to regard the motion as sensibly irrotational at the boundary $r = \mathrm{R}$.

Now, at the external boundary $r = \mathrm{R}$ we may write approximately

$$\mathrm{R}\phi_1 = \sum_{n=0}^{\infty} [\mathrm{A}_n h^{-(n+1)} . \iota^n \mathrm{P}_n(\mu)\, e^{\iota(\sigma t - h\mathrm{R})}]$$

where square brackets are used to denote that the real part only of the expression so enclosed is to be taken into the account.

Hence we obtain

$$\mathrm{R}p_1 = -\rho_0\sigma \sum_{n=0}^{\infty} [\mathrm{A}_n h^{-(n+1)} \iota^{n+1} \mathrm{P}_n(\mu) . e^{\iota(\sigma t - h\mathrm{R})}], \qquad \mathrm{R}q_1 = h \sum_{n=0}^{\infty} [\mathrm{A}_n h^{-(n+1)} \iota^{n+1} . \mathrm{P}_n(\mu)\, e^{\iota(\sigma t - h\mathrm{R})}].$$

Combining these results we find

$$\mathrm{R}(p_0 q_1 + p_1 q_0) = (-\rho_0\sigma q_0 + h p_0) \sum_{n=0}^{\infty} [\mathrm{A}_n h^{-(n+1)} . \iota^{n+1} \mathrm{P}_n(\mu)\, e^{\iota(\sigma t - h\mathrm{R})}] \quad . \quad . \quad . \quad (2).$$

Again we have from (3) § 7, since $h\mathrm{R}$ is large,

$$\mathrm{R}\phi_0 = \sum_{n=0}^{\infty} \{(2n+1) h^{-1} \mathrm{P}_n(\mu) \sin(h\mathrm{R} - \tfrac{1}{2}n\pi) \cos(\sigma t + \tfrac{1}{2}n\pi)\}.$$

Hence we have

$$\mathrm{R}p_0 = \rho_0\sigma \sum_{n=0}^{\infty} \{(2n+1) h^{-1} \mathrm{P}_n(\mu) \sin(h\mathrm{R} - \tfrac{1}{2}n\pi) \sin(\sigma t + \tfrac{1}{2}n\pi)\},$$

$$\mathrm{R}q_0 = -h \sum_{n=0}^{\infty} \{(2n+1) h^{-1} \mathrm{P}_n(\mu) \cos(h\mathrm{R} - \tfrac{1}{2}n\pi) \cos(\sigma t + \tfrac{1}{2}n\pi)\}.$$

Combining the two last results we find

$$\mathrm{R}(-\rho_0\sigma q_0 + h p_0) = \rho_0\sigma \sum_{n=0}^{\infty} \{(-)^n (2n+1) \mathrm{P}_n(\mu) \cos(\sigma t - h\mathrm{R})\}.$$

Substituting this result in (2) and integrating over the surface of the sphere $r = \mathrm{R}$, we obtain

$$\iint (p_1 q_0 + p_0 q_1)\, d\mathrm{S} = 4\pi\rho_0\sigma \sum_{n=0}^{\infty} [(-)^n \mathrm{A}_n \iota^{n+1} h^{-(n+1)} \cos(\sigma t - h\mathrm{R})\, e^{\iota(\sigma t - h\mathrm{R})}],$$

of which the mean value is

$$2\pi\rho_0\sigma \sum_{n=0}^{\infty} [(-)^n \mathrm{A}_n \iota^{n+1} h^{-(n+1)}] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

This last expression then represents the loss of energy to the primary waves in consequence of the presence of the obstacle. From the value of A_n obtained in (15), § 7, we see that the summation (3) consists of a series of terms arranged in descending order of magnitude. Consequently, in determining its value we may limit our attention to the first two terms. Hence the total loss of energy to the primary waves is given very approximately by

$$2\pi\rho_0\sigma h [\mathrm{A}_0 \iota h^{-2} + \mathrm{A}_1 h^{-3}] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

Now, from (17) and (18), § 7, we have

$$[\mathrm{A}_0 \iota h^{-2} + \mathrm{A}_1 h^{-3}] = \tfrac{1}{4} h a^3 \{3\sqrt{2}\,.\,\lambda^{-1}a^{-1} + 6\lambda^{-2}a^{-2} + 3h^2a^2(\lambda^{-2}a^{-2} + \sqrt{2}\lambda^{-3}a^{-3}) + h^3a^3(\tfrac{7}{9} + \sqrt{2}\lambda^{-1}a^{-1} + 3\sqrt{2}\lambda^{-2}a^{-2} - 3\lambda^{-4}a^{-4})\} \quad . \quad . \quad . \quad (5).$$

Let us first consider the case when λa is small. Since h^2/λ^2 or $\sigma\nu/c^2$ is always small, it follows from (5) that in this case we may write approximately

$$[\mathrm{A}_0 \iota h^{-2} + \mathrm{A}_1 h^{-3}] = \tfrac{1}{4} a^2 \left(6\frac{\nu}{ca} + 3\sqrt{2}\sigma^{1/2}\nu^{1/2}/c\right).$$

When λa is great, it is necessary to include one other term of (5), and we may in general write in this case

$$[\mathrm{A}_0 \iota h^{-2} + \mathrm{A}_1 h^{-3}] = \tfrac{1}{4} a^2 \left(\tfrac{7}{9}\sigma^4 a^4/c^4 + 3\sqrt{2}\sigma^{1/2}\nu^{1/2}/c + 6\frac{\nu}{ca}\right) \quad . \quad . \quad . \quad . \quad (6).$$

Comparing this last result with that obtained in the case when λa is small, we see that we may take it as a sufficient approximation in almost all cases. For small values of the radius the first term in (6) will be negligible.

Substituting from (6) in (4) we obtain for the total loss of energy to the primary waves the expression

$$\tfrac{1}{2}\rho_0\sigma^2/c\pi a^2\,.\left(\tfrac{7}{9}\sigma^4 a^4/c^4 + 3\sqrt{2}\sigma^{1/2}\nu^{1/2}/c + 6\frac{\nu}{ca}\right).$$

Now the energy incident upon the obstacle in the primary waves is given by $\frac{1}{2}\rho_0\sigma^2/c\pi a^2$, and hence the ratio of the lost energy to that incident upon the obstacle is

$$3\sqrt{2}\sigma^{1/2}\nu^{1/2}/c + 6\frac{\nu}{ca} + \tfrac{7}{9}\sigma^4 a^4/c^4 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7).$$

The first two terms of this last expression represent the proportion of the incident energy lost by friction. The last term of (7) gives the proportion lost by scattering to a distance, and is the same as is obtained in the theory of a frictionless air.

When λa is small, the most important term of (7) is the second $6\frac{\nu}{ca}$. Hence we see that in the case of small obstacles the ratio of the lost energy to the incident energy varies inversely as the radius of the obstacle, and consequently tends to become very great as this radius is diminished. On the other hand, the actual amount of energy lost varies directly as the radius of the obstacle, and diminishes with the radius. It is to be noticed that in the case of sufficiently small obstacles the energy lost to the primary waves is independent of the wave-length of the incident sound.

When λa is great, the most important term of (7) is the first $3\sqrt{2}\sigma^{1/2}\nu^{1/2}/c$. Hence we see that in this case the ratio of the lost energy to that incident upon the obstacle is very nearly independent of the radius of the obstacle, provided the order of magnitude of this ratio is altered by the viscosity. Consequently for sufficiently large obstacles the loss of energy to the primary waves is proportional to the surface of the obstructing sphere. There is, it will be noticed, a distinct similarity between these results and those obtained above in the case of cylindrical obstacles.

The expression (7) has been evaluated in a number of different cases, and the results are arranged on p. 266 in tabular form. K denotes the expression (7) or the ratio of the lost energy to that incident upon the obstacle, and λ represents the wave-length (measured in centimetres) of the incident sound.

§ 9. *Application of the above to the Problem of a Large Number of Spherical Obstacles.*—Let us consider now the loss of energy to the primary waves when these are incident upon a large number of spherical obstacles. We shall suppose that there are n small spheres per c.cm.; the validity of our argument will depend on the volume occupied by the obstacles being small compared with the total volume. Consequently $\frac{4}{3}n\pi a^3$ must be a small fraction.

At a distance r from the centre of any one of these spherical particles, great compared with the wave-length of the incident sound, the secondary waves due to that particle will be sensibly irrotational, and will be given very approximately in all cases by

$$\phi_1 = \mathrm{A}_0 f_0(hr) + \mathrm{A}_1 f_1(hr)\, r\mu.$$

Since hr is great, we may write

$$f_0(hr) = \frac{e^{-\iota hr}}{hr}, \qquad f_1(hr) = \iota\frac{e^{-\iota hr}}{h^2r^2},$$

and ϕ_1 takes the form

$$\phi_1 = (\mathrm{A}_0 h^{-1} + \mathrm{A}_1 h^{-2}\iota\mu)\, e^{-\iota hr}/r\,;$$

which, along the course of the primary waves ($\mu = -1$), reduces to

$$\phi_1 = (\mathrm{A}_0 h^{-1} - \mathrm{A}_1 h^{-2}\iota)\, e^{-\iota hr}/r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

Consider now the spheres which occupy a thin stratum dx perpendicular to the course of the primary waves. Let P be any point in this stratum, and let O be the point where the vibration is to be estimated at a great distance from the stratum.*

If AP $= z$, the element of volume is $2\pi x.dx.dz$, and consequently the number of spherical particles in it is $2\pi nz.dx.dz$. Also, if OP $= r$, AO $= -x$, then $r^2 = x^2 + z^2$ and $r\,dr = z\,dz$.

Now by (1) the resultant at O of all the secondary vibrations which issue from the stratum is given by

$$2\pi n\, dx \int_{-x}^{\infty} (\mathrm{A}_0 h^{-1} - \mathrm{A}_1 h^{-2}\iota)\, e^{-\iota hr}\, dr.$$

Remembering that the angle AOP is to be regarded as very small, we see that the

---

* See figure, p. 254.

a = 10 cm.		a = 1 cm.		a = ·1 cm.		a = ·01 cm.		a = ·001 cm.	
λ.	K.	λ.	K.	λ.	K.	λ.	K.	λ.	K.
300	2·7 10^{-3}	30	5·4 10^{-3}	5	9·9 10^{-3}	5	11·9 10^{-3}	5	3·3 10^{-2}
400	1·5 10^{-3}	40	3·9 10^{-3}	10	7·0 10^{-3}	10	9·0 10^{-3}	10	3·1 10^{-2}
500	1·1 10^{-3}	50	3·2 10^{-3}	20	5·0 10^{-3}	20	7·1 10^{-3}	20	2·9 10^{-2}
600	·9 10^{-3}	60	2·9 10^{-3}	30	4·1 10^{-3}	30	6·3 10^{-3}	30	2·8 10^{-2}
700	·9 10^{-3}	70	2·7 10^{-3}	40	3·6 10^{-3}	40	5·7 10^{-3}	40	2·7 10^{-2}
800	·8 10^{-3}	80	2·4 10^{-3}	50	3·2 10^{-3}	50	5·4 10^{-3}	50	2·7 10^{-2}
900	·7 10^{-3}	90	2·3 10^{-3}	60	3·0 10^{-3}	60	5·1 10^{-3}	60	2·7 10^{-2}
1000	·7 10^{-3}	100	2·1 10^{-3}	70	2·8 10^{-3}	70	4·8 10^{-3}	70	2·6 10^{-2}
		200	1·5 10^{-3}	80	2·6 10^{-3}	80	4·8 10^{-3}	80	2·6 10^{-2}
		300	1·2 10^{-3}	90	2·5 10^{-3}	90	4·6 10^{-3}	90	2·6 10^{-2}
		400	1·1 10^{-3}	100	2·4 10^{-3}	100	4·5 10^{-3}	100	2·6 10^{-2}
		500	1·0 10^{-3}	200	1·7 10^{-3}	200	3·9 10^{-3}	200	2·5 10^{-2}
		600	·9 10^{-3}	300	1·5 10^{-3}	300	3·6 10^{-3}	300	2·5 10^{-2}
		700	·8 10^{-3}	400	1·3 10^{-3}	400	3·4 10^{-3}	400	2·5 10^{-2}
		800	·8 10^{-3}	500	1·2 10^{-3}	500	3·3 10^{-3}	500	2·5 10^{-2}
		900	·7 10^{-3}	600	1·1 10^{-3}	600	3·3 10^{-3}	600	2·5 10^{-2}
		1000	·7 10^{-3}	700	1·0 10^{-3}	700	3·2 10^{-3}	700	2·5 10^{-2}
				800	1·0 10^{-3}	800	3·1 10^{-3}	800	2·5 10^{-2}
				900	·9 10^{-3}	900	3·1 10^{-3}	900	2·5 10^{-2}
				1000	·9 10^{-3}	1000	3·1 10^{-3}	1000	2·5 10^{-2}

resultant at O of all the secondary vibrations coming from the stratum dx is given by

$$-2\pi n\,dx\,(\iota A_0 h^{-2}+A_1 h^{-3})\,e^{\iota(hx\sigma t)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2),$$

where the time factor $e^{\iota\sigma t}$ has been restored.

Now, from the results obtained in § 7 for A_1 and A_0, we find

$$\begin{aligned}\iota A_0 h^{-2}+A_1 h^{-3} = {} & \tfrac{1}{4}ha^3\{3\sqrt{2}\lambda^{-1}a^{-1}+6\lambda^{-2}a^{-2}+3h^2a^2(\lambda^{-2}a^{-2}+\sqrt{2}\lambda^{-3}a^{-3}) \\ & \qquad +h^3a^3(\tfrac{7}{9}+\sqrt{2}\lambda^{-1}a^{-1}+3\sqrt{2}\lambda^{-3}a^{-3}-3\lambda^{-4}a^{-4})\} \\ & +\tfrac{1}{2}\iota ha^3\{\tfrac{1}{3}+\tfrac{3}{2}\sqrt{2}\lambda^{-1}a^{-1}+\tfrac{1}{2}h^2a^2(\tfrac{1}{5}-3\sqrt{2}\lambda^{-3}a^{-3}+3\lambda^{-4}a^{-4}) \\ & \qquad -\tfrac{1}{2}h^3a^3(\sqrt{2}\lambda^{-1}a^{-1}+5\lambda^{-2}a^{-2}-3\sqrt{2}\lambda^{-3}a^{-3})\}.\end{aligned}$$

Except for very minute obstacles, it will be sufficient to write

$$\iota A_0 h^{-2}+A_1 h^{-3} = \tfrac{1}{4}ha^3\{3\sqrt{2}\lambda^{-1}a^{-1}+6\lambda^{-2}a^{-2}+\tfrac{7}{9}h^3a^3\}+\tfrac{1}{2}\iota ha^3(\tfrac{1}{3}+\tfrac{3}{2}\sqrt{2}\lambda^{-1}a^{-1}).$$

Substituting this last expression in (2), we obtain for the resultant of all the secondary vibrations coming from the stratum dx

$$-\tfrac{1}{2}n\pi\,dx\sigma a^3/c\,[(3\sqrt{2}\lambda^{-1}a^{-1}+6\lambda^{-2}a^{-2}+\tfrac{7}{9}h^3a^3)+\iota(\tfrac{2}{3}+3\sqrt{2}\lambda^{-1}a^{-1})]\,e^{\iota(hx+\sigma t)},$$

of which the real part is

$$\begin{aligned}-\tfrac{1}{2}n\pi\,dx.\sigma a^3/c\,\{ & (3\sqrt{2}\lambda^{-1}a^{-1}+6\lambda^{-2}a^{-2}+\tfrac{7}{9}h^3a^3)\cos(hx+\sigma t) \\ & -(\tfrac{2}{3}+3\sqrt{2}\lambda^{-1}a^{-1})\sin(hx+\sigma t)\} \quad . \qquad (3).\end{aligned}$$

To this is to be added the corresponding expression for the primary wave

$$\phi_0 = \cos(hx+\sigma t).$$

The coefficient of $\cos(hx+\sigma t)$ is thus altered by the obstacles in the layer dx from unity to

$$1-\tfrac{1}{2}n\pi a^2\left\{6\frac{\nu}{ca}+3\sqrt{2}\sigma^{1/2}\nu^{1/2}/c+\tfrac{7}{9}\sigma^4a^4/c^4\right\}dx.$$

Thus, if E be the energy in the incident wave, we have

$$dE/E = -n\pi a^2\left\{6\frac{\nu}{ca}+3\sqrt{2}\sigma^{1/2}\nu^{1/2}/c+\tfrac{7}{9}\sigma^4a^4/c^4\right\}dx.$$

Integrating this, we obtain

$$E = E_0 e^{-\alpha x},$$

where E_0 is the energy in the primary waves at incidence, and α is given by

$$\alpha = n\pi a^2\left(6\frac{\nu}{ca}+3\sqrt{2}\sigma^{1/2}\nu^{1/2}/c+\tfrac{7}{9}\sigma^4a^4/c^4\right) \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4).$$

If the radius of each obstacle is measured in centimetres, then the reciprocal of α, as determined by (4), will give the distance travelled by the sound before its intensity is diminished in the ratio of $1/e$. If the radius of each small sphere is 10^{-3} cm., and there are 10^6 per c.cm., then $\frac{4}{3}n\pi a^3$ will be a small fraction, and the formula (4) will be applicable. With these numerical values we obtain, in the case of sound of wave-length 50 cm., $\alpha = 8{\cdot}5\ 10^{-2}$. Consequently $a^{-1} = 11{\cdot}8$ cm.; hence, after passing through a thickness of less than 12 cm. of such a medium, the intensity of the sound will be diminished in the ratio of $1/e$.

The formula (4) should be applicable to fogs, as we may regard the water particles as approximately fixed, since their inertia is so much greater than that of the surrounding air. I am indebted to Prof. LAMB for the following information from HANN'S 'Meteorologie': "In a dense fog the amount of water may vary from about 3 to 10 gr. per cubic metre. Assuming that the diameter of the drops is ·02 mm., and a cubic metre contains 4·5 gr. of water, this is calculated to give 10^9 drops per cubic metre, and therefore 10^3 per cubic centimetre." With these numerical data the formula (4) gives $\alpha^{-1} = 1180$ metres, and consequently it follows that the fog would not interfere appreciably with the propagation of sound. But if the diameter of the drops could be as small as ·002 mm., a fog of the same density would contain 10^6 drops per cubic centimetre, and α^{-1} would be nearly equal to $1\frac{1}{3}$ metres, and consequently the sound would be damped very quickly by the fog.* On the other hand, TYNDALL'S observations appear to show that the presence of fog is not prejudicial to the audibility of sound.†

The coefficient of sin $(hx+\sigma t)$ in (3) gives the refractivity of the medium as modified by the spherical particles. If δ be the retardation due to the spheres of the stratum dx,

$$\delta = \tfrac{1}{2}n\pi\,.\,dx\,.\,a^3\left(\tfrac{2}{3}+3\sqrt{2}\,\lambda^{-1}a^{-1}\right).$$

Hence, if μ be the refractive index of the medium as modified by the particles,

$$\mu-1 = \tfrac{3}{8}p\left(\tfrac{2}{3}+3\sqrt{2}\,\lambda^{-1}a^{-1}\right),$$

where p denotes the ratio, *assumed small*, of the volume occupied by the particles to the total volume.

Hence finally we have

$$\mu-1 = p\left\{\tfrac{1}{4}+\tfrac{9}{8}\sqrt{2}\,(\nu/\sigma)^{\frac{1}{2}}/a\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

For sound of wave-length 50 cm. incident upon a medium in which there are 10^6 spherical particles per cubic centimetre, each of radius 10^{-3} cm., we obtain

$$\mu-1 = 3\,.\,7\,.\,10^{-2}.$$

---

* See, however, note at end.

† RAYLEIGH, 'Treatise on Sound,' Vol. II., p. 137.

Note, April 12th, 1910.

Prof. LARMOR has kindly pointed out to me that it is not legitimate to apply the formula (4) to fogs without further consideration. Although the inertia of the water particles is so much greater than that of the surrounding air, yet in consequence of the viscosity of the air it does not follow that we may regard the water particles as approximately fixed. I have investigated the problem of a free spherical obstacle. The analysis is very similar to that in the problem of the fixed obstacle. The secondary waves diverging from the obstacle are affected only in the terms containing spherical harmonics of the first order. If $Ue^{\iota\sigma t}$ is the velocity of the obstacle along the axis of x we obtain

$$\tfrac{1}{3}A_1 f_0(ha) + B_1 2 f_0(ka) = -\iota h\psi_0(ha) + U \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

$$\tfrac{1}{3}A_1 h^2a^2 f_2(ha) - B_1 k^2a^2 f_2(ka) = -\iota h h^2a^2\psi_2(ha) \quad . \quad . \quad . \quad . \quad . \quad (2),$$

together with

$$M\iota\sigma U = \iint p_{rx}\, dS \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where p_{rx} is the component of the stress across the surface of the obstacle in the direction of the axis of x and the integration is taken over the surface of the obstacle.

The last equation reduces to

$$U = \frac{\rho_0}{\rho_1}\left[U + 3B_1\{3f_1(ka) - f_0(ka)\}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where ρ_1 is the density of the obstacle.

Eliminating A_1 and B_1 between the equations (1), (2), and (4) we obtain approximately when ka is small

$$U\left\{1 - \tfrac{9}{2}\frac{\rho_0}{\rho_1} \,.\, k^{-2}a^{-2}\right\} = -\tfrac{9}{2}\iota h\frac{\rho_0}{\rho_1}k^{-2}a^{-2}.$$

Hence, if L be the ratio of the amplitude of the motion of the obstacle to that of the waves of sound, we have

$$L = \{1 + \tfrac{4}{81}(\rho_1/\rho_0)^2\lambda^{-4}a^{-4}\}^{-1/2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

Again, eliminating B_1 and U from (1), (2), and (4) we obtain approximately

$$A_1 = -\tfrac{1}{2}\iota h^4a^3k^2a^2 f_2(ka)\left[f_0(ka) - \tfrac{3}{2}\frac{\rho_0}{\rho_1}\{3f_1(ka) - f_0(ka)\}\right]^{-1}.$$

Hence we obtain without difficulty when ka is small

$$[A_1] = [A_1]_0\{1 + \tfrac{81}{4}(\rho_0/\rho_1)^2\lambda^{-4}a^{-4}\}^{-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6),$$

where $[A_1]$, $[A_1]_0$ denote the real part of the value of A in the case of the fixed and free obstacle.

The last relation may be written in the form

$$[A_1] = [A_1]_0 (1 - L^2) \qquad \qquad (7).$$

It follows that the ratio of the lost energy to that incident upon the obstacle is given by

$$\tfrac{7}{9}\sigma^4 a^4/c^4 + \left(3\sqrt{2}\,\sigma^{1/2} v^{1/2}/c + 6\frac{v}{ca}\right)(1 - L^2) \qquad \qquad (8).$$

Extending this result to the case of a number of free spherical obstacles we obtain, instead of (4), § 9

$$\alpha = n\pi a^2 K$$

where K represents the ratio of the lost to the incident energy given in (8).

It follows from this investigation that the results obtained in this paper are only applicable to fogs if $\frac{9}{2}(\rho_0/\rho_1)\lambda^{-2}a^{-2}$ is a small fraction. This condition is satisfied for obstacles of radius 10^{-2} cm., and also for obstacles of radius 10^{-3} cm. when the wave-length of the incident sound is not too long. In the case of obstacles of radius 10^{-4} cm., however, this condition is no longer satisfied; L approaches close to unity for all wave-lengths, and consequently α and K are very small. Hence, if the diameter of the drops of water in a fog is as small as ·002 mm., such a fog does not interfere appreciably with the propagation of sound, and a result is obtained in agreement with TYNDALL's observations.

I append a table giving the values of L in a number of different cases. When the wave-length of the sound is very great, or when the obstacle is extremely minute, the obstacle vibrates with the air surrounding it.

$a = 10^{-3}$ cm.				$a = 10^{-4}$ cm.	
λ.	L.	λ.	L.	λ.	L.
5	·019	200	·595	5	·877
10	·037	300	·746	10	·961
20	·074	400	·833	20	·990
30	·110	500	·877	30	·996
40	·147	600	·918	100	·9996
50	·182	700	·935	∞	1·000
60	·217	800	·955		
70	·251	900	·961		
80	·284	1000	·971		
90	·316	∞	1·000		
100	·347				